PART I

# Introduction to Reinforcement Learning

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## What is Machine Learning

A more mathematical definition by Tom Mitchell

- Machine learning is the study of algorithms that
  - improve their performance *P*
  - at some task *T*
  - based on experience *E*
  - with non-explicit programming

- A well-defined learning task is given by <*P, T, E*>

## Supervised Learning

- Given the training dataset of (data, label) pairs,

$$D = \{(x_i, y_i)\}_{i=1,2,\ldots,N}$$

  let the machine learn a function from data to label

$$y_i \simeq f_\theta(x_i)$$

- Learning is referred to as updating the parameter $\theta$
- Learning objective: make the prediction close to the ground truth

$$\min_\theta \frac{1}{N} \sum_{i=1}^{N} \mathcal{L}(y_i, f_\theta(x_i))$$

## Unsupervised Learning

- Given the training dataset

$$D = \{x_i\}_{i=1,2,\ldots,N}$$

let the machine learn the data underlying patterns

- Sometimes build latent variables

$$z \rightarrow x$$

- Estimate the probabilistic density function (p.d.f.)

$$p(x;\theta) = \sum_z p(x|z;\theta)p(z;\theta)$$

- Maximize the log-likelihood of training data

$$\max_\theta \frac{1}{N} \sum_{i=1}^{N} \log p(x;\theta)$$

## Two Kinds of Machine Learning

- Prediction
  - Predict the desired output given the data (supervised learning)
  - Generate data instances (unsupervised learning)
  - We mainly covered this category in previous lectures

- Decision Making
  - Take actions based on a particular state in a dynamic environment (reinforcement learning)
    - to transit to new states
    - to receive immediate reward
    - to maximize the accumulative reward over time
  - Learning from interaction

# Machine Learning Categories

- Supervised Learning
  - To perform the desired output given the data and labels

$$p(y|x)$$

- Unsupervised Learning
  - To analyze and make use of the underlying data patterns/structures

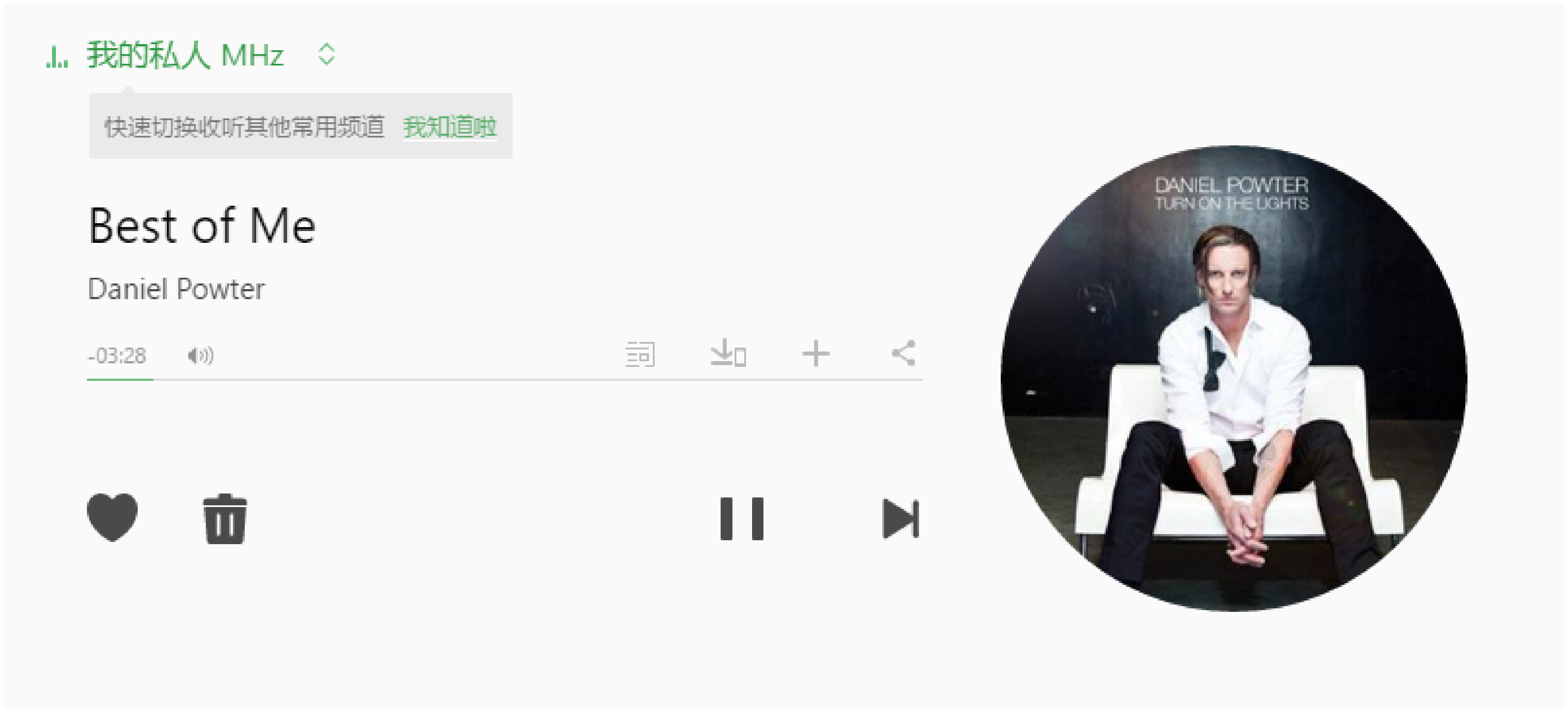
$$p(x)$$

- Reinforcement Learning
  - To learn a policy of taking actions in a dynamic environment and acquire rewards

$$\pi(a|x)$$

## RL Use Case 1: Interactive Recommendation

- Douban.fm music recommend and feedback
  - The machine needs to make decisions, not just prediction



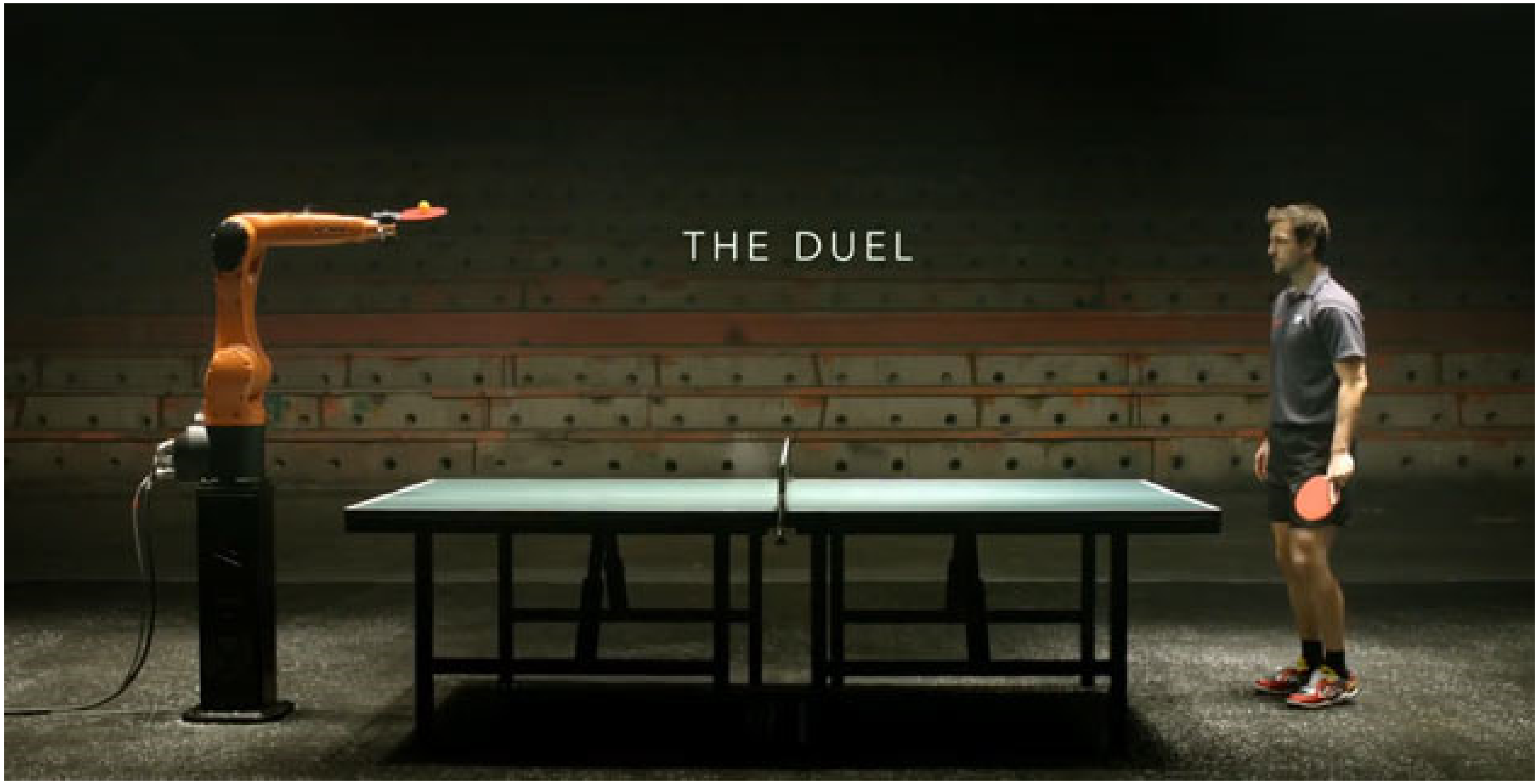
Xiaoxue Zhao, Weinan Zhang et al. Interactive Collaborative Filtering. CIKM 2013.

# RL Use Case 2: Robotics Control

- Stanford Autonomous Helicopter
  - http://heli.stanford.edu/

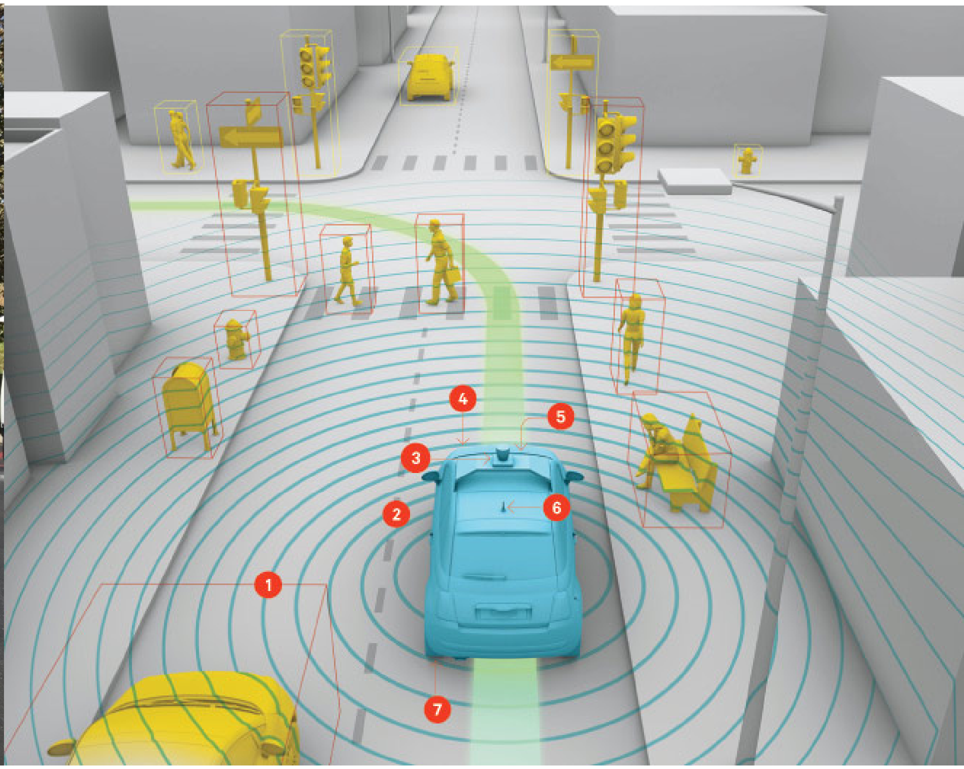
## RL Use Case 3: Robotics Control

- Ping pong robot
  - https://www.youtube.com/watch?v=tIIJME8-au8



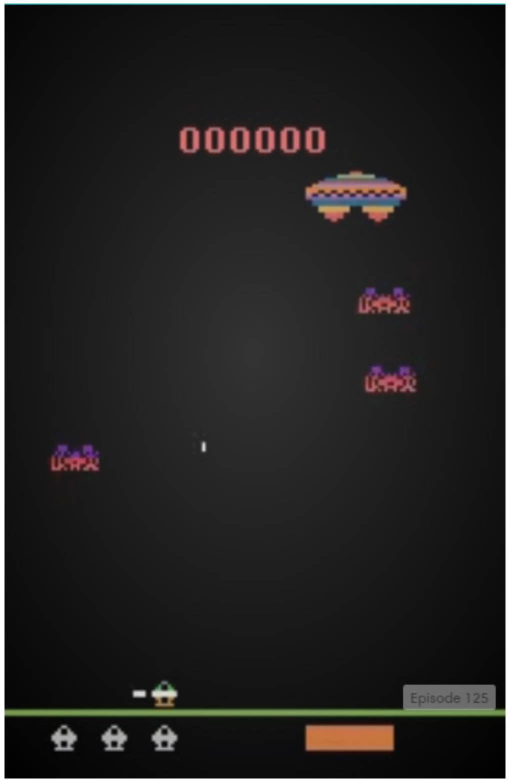
# RL Use Case 4: Self-Driving Cars

- Google Self-Driving Cars
  - https://www.google.com/selfdrivingcar/



# RL Use Case 5: Game Playing

- Take actions given screen pixels
    - https://gym.openai.com/envs#atari



Mnih, Volodymyr, et al. "Human-level control through deep reinforcement learning." *Nature* 518.7540 (2015): 529-533.

# Reinforcement Learning Materials

Our lecture on RL is mainly based on the materials from these masters.

**Prof. Richard Sutton**

- University of Alberta, Canada
- http://incompleteideas.net/sutton/index.html
- Reinforcement Learning: An Introduction (2nd edition)
- http://www.incompleteideas.net/book/the-book-2nd.html

**Prof. David Silver**

- Google DeepMind and UCL, UK
- http://www0.cs.ucl.ac.uk/staff/d.silver/web/Home.html
- UCL Reinforcement Learning Course
- http://www0.cs.ucl.ac.uk/staff/d.silver/web/Teaching.html

**Prof. Andrew Ng**

- Stanford University, US
- http://www.andrewng.org/
- Machine Learning (CS229) Lecture Notes 12: RL
- http://cs229.stanford.edu/materials.html

## Content

- Introduction to Reinforcement Learning
- Model-based Reinforcement Learning
  - Markov Decision Process
  - Planning by Dynamic Programming
- Model-free Reinforcement Learning
  - On-policy SARSA
  - Off-policy Q-learning
  - Model-free Prediction and Control

## Content

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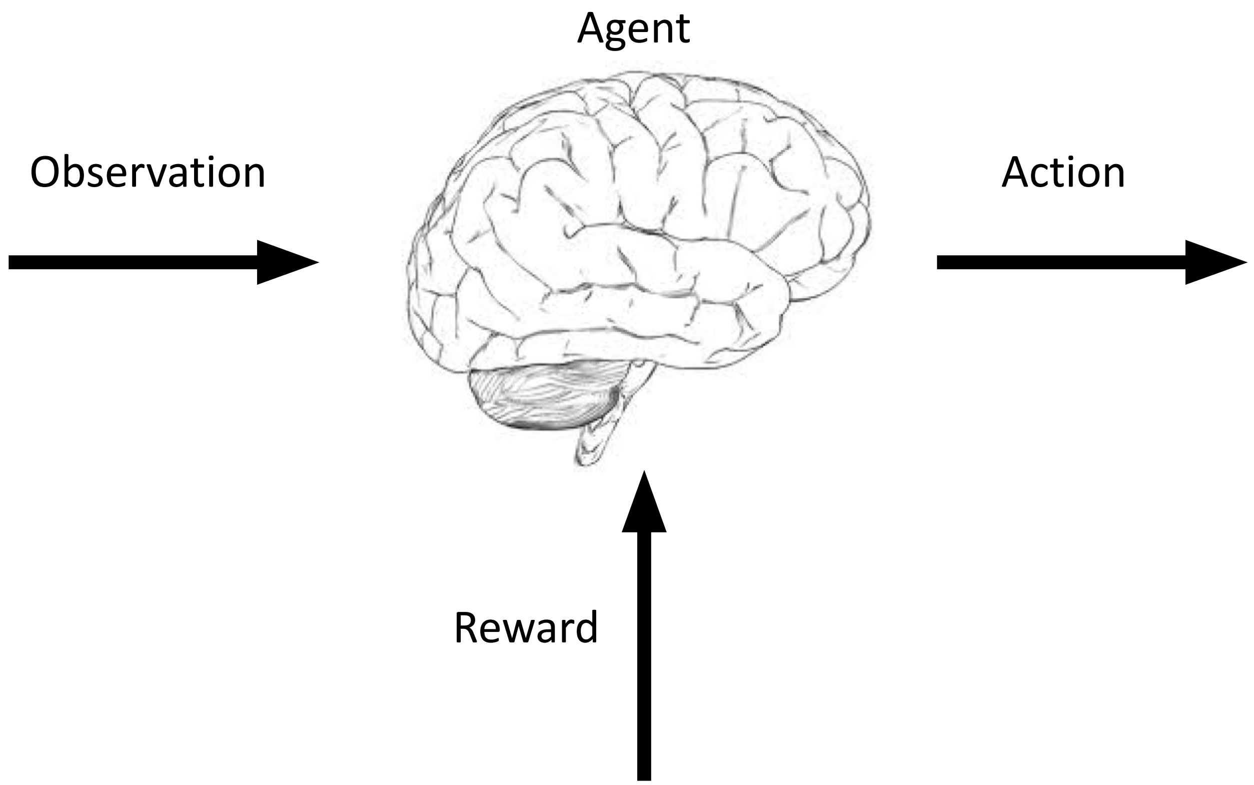
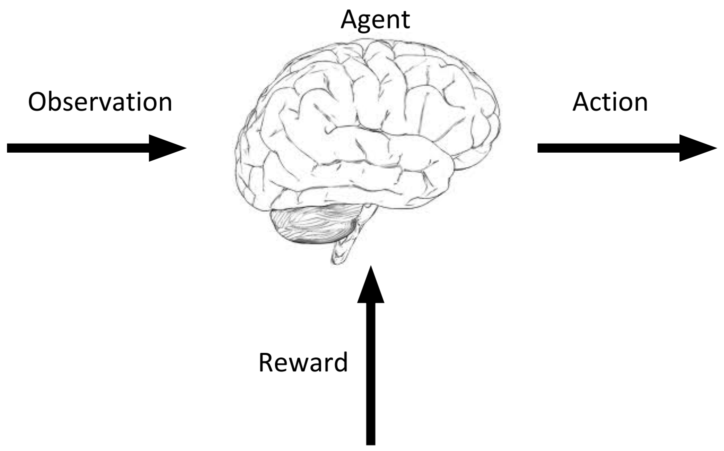
## Reinforcement Learning

- Learning from interaction
  - Given the current situation, what to do next in order to maximize utility?



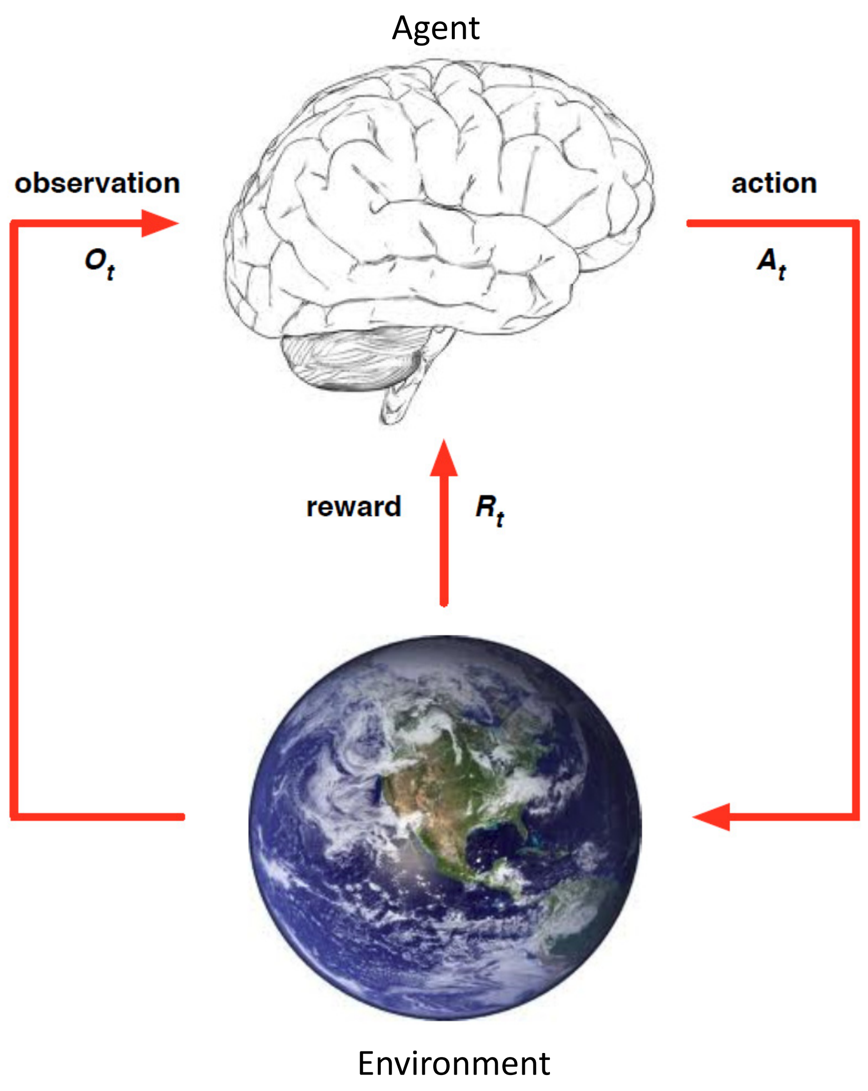
# Reinforcement Learning Definition

- A computational approach by learning from interaction to achieve a goal



- Three aspects
  - Sensation: sense the state of the environment to some extent
  - Action: able to take actions that affect the state and achieve the goal
  - Goal: maximize the cumulative reward over time

# Reinforcement Learning



- At each step $t$, the agent
  - Receives observation $O_t$
  - Receives scalar reward $R_t$
  - Executes action $A_t$
- The environment
  - Receives action $A_t$
  - Emits observation $O_{t+1}$
  - Emits scalar reward $R_{t+1}$
- $t$ increments at environment step

# Elements of RL Systems

- History is the sequence of observations, action, rewards

$$H_t = O_1, R_1, A_1, O_2, R_2, A_2, \dots, O_{t-1}, R_{t-1}, A_{t-1}, O_t, R_t$$

  - i.e. all observable variables up to time *t*
  - E.g., the sensorimotor stream of a robot or embodied agent
- What happens next depends on the history:
  - The agent selects actions
  - The environment selects observations/rewards
- State is the information used to determine what happens next (actions, observations, rewards)
- Formally, state is a function of the history

$$S_t = f(H_t)$$

# Elements of RL Systems

- Policy is the learning agent's way of behaving at a given time
    - It is a map from state to action
    - Deterministic policy

$$a = \pi(s)$$

    - Stochastic policy

$$\pi(a|s) = P(A_t = a|S_t = s)$$

## Elements of RL Systems

- Reward
  - A scalar defining the goal in an RL problem
  - For immediate sense of what is good

- Value function
  - State value is a scalar specifying what is good in the long run
  - Value function is a prediction of the cumulative future reward
    - Used to evaluate the goodness/badness of states (given the current policy)

$$v_\pi(s) = \mathbb{E}_\pi[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \dots | S_t = s]$$

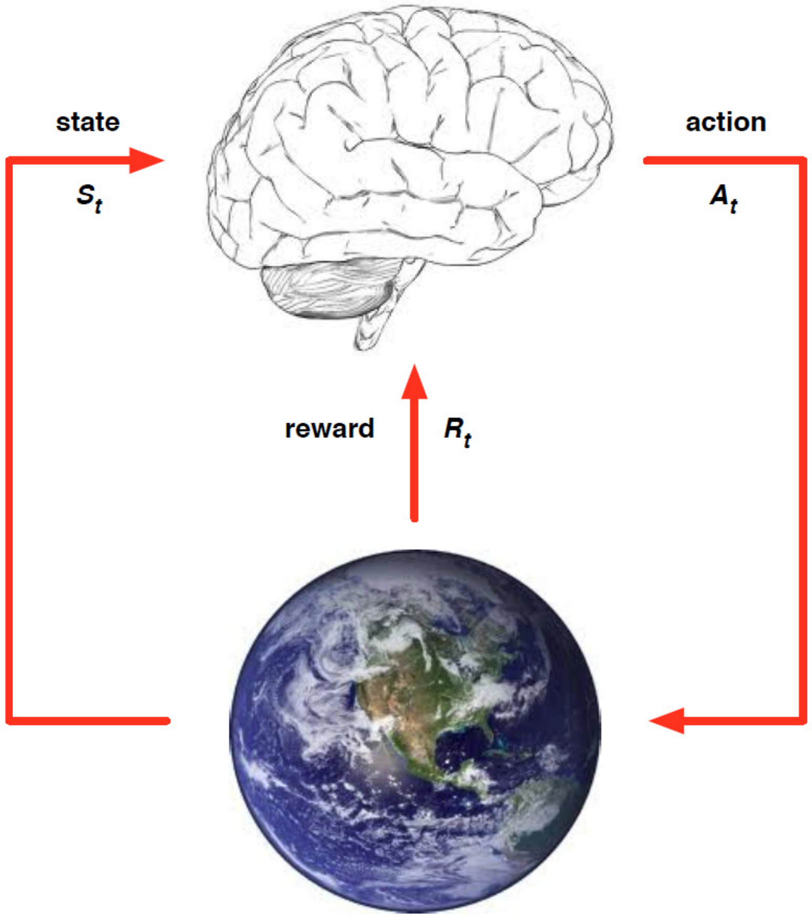
# Elements of RL Systems

- A Model of the environment that mimics the behavior of the environment
  - Predict the next state

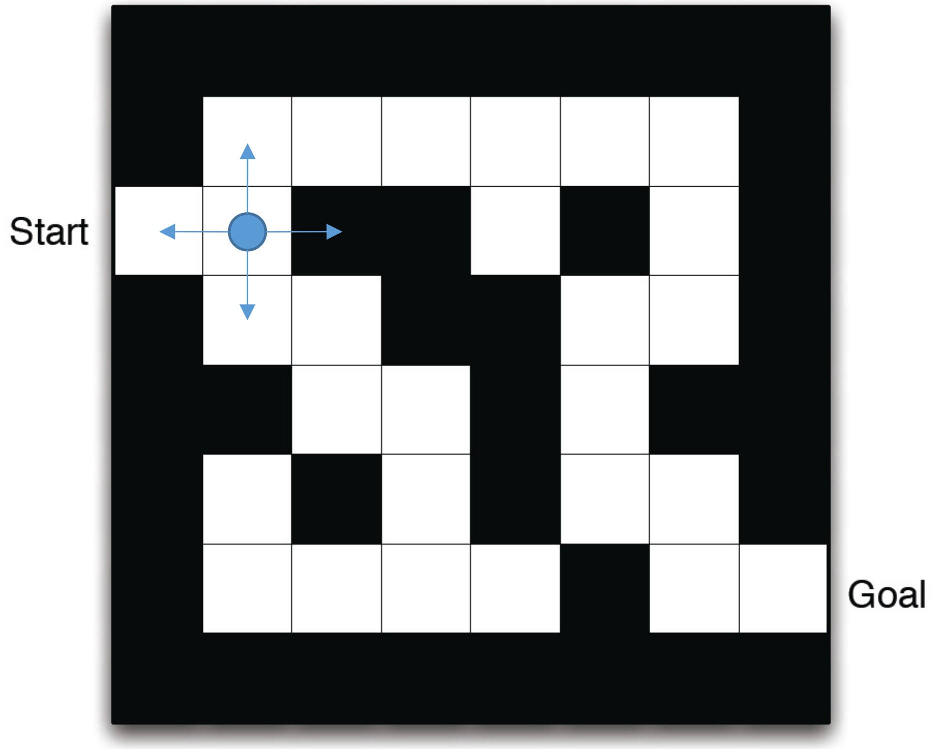
$$\mathcal{P}_{ss'}^{a} = \mathbb{P}[S_{t+1} = s' | S_t = s, A_t = a]$$

  - Predicts the next (immediate) reward

$$\mathcal{R}_{s}^{a} = \mathbb{E}[R_{t+1} | S_t = s, A_t = a]$$

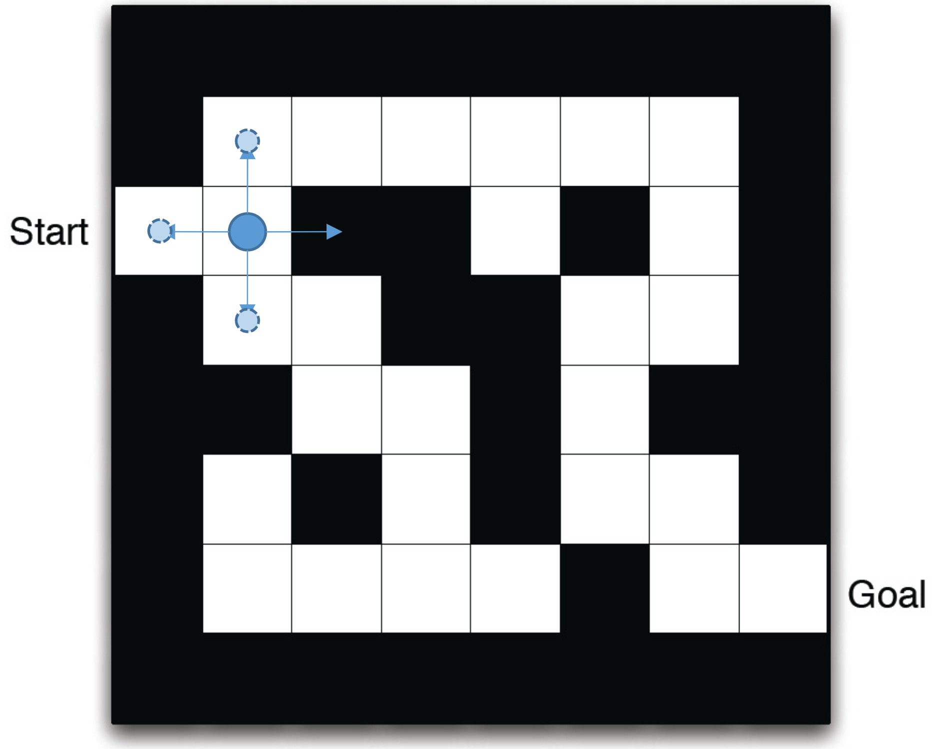


# Maze Example



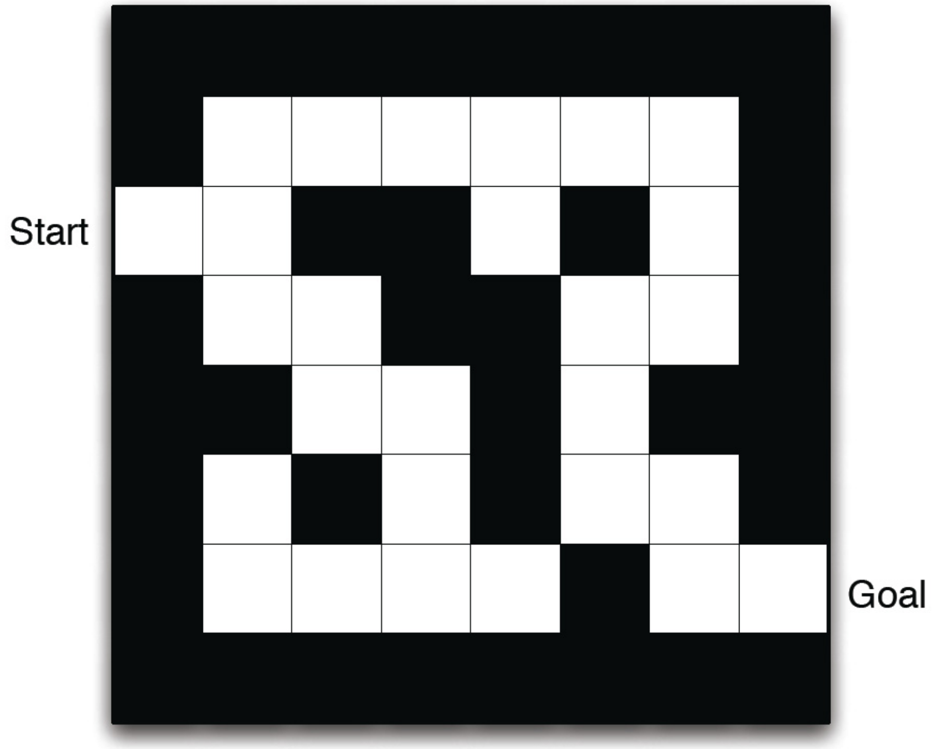
- State: agent's location
- Action: N,E,S,W

# Maze Example



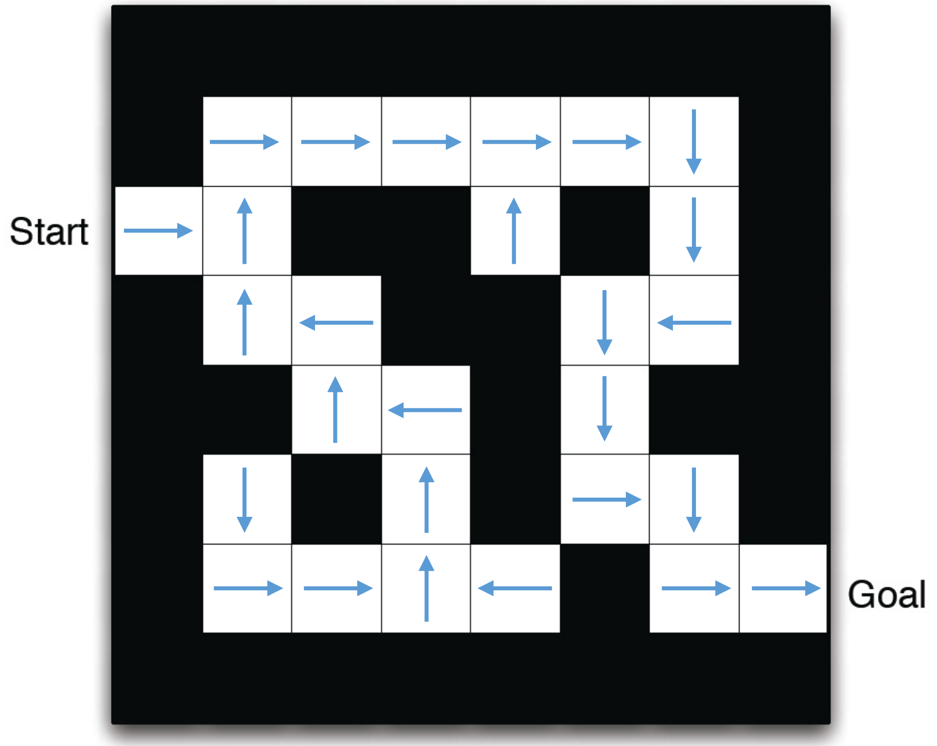
- State: agent's location
- Action: N,E,S,W
- State transition: move to the next grid according to the action
    - No move if the action is to the wall

## Maze Example



- State: agent's location
- Action: N,E,S,W
- State transition: move to the next grid according to the action
- Reward: -1 per time step

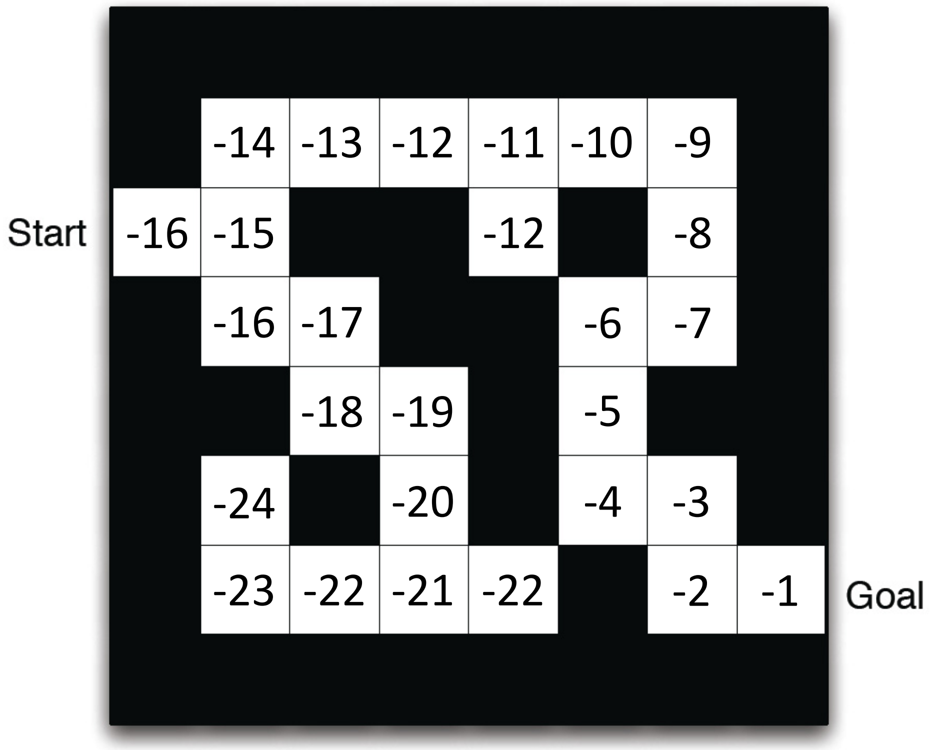
# Maze Example



- State: agent's location
- Action: N,E,S,W
- State transition: move to the next grid according to the action
- Reward: -1 per time step

- Given a policy as shown above
  - Arrows represent policy $\pi(s)$ for each state $s$

# Maze Example



- State: agent’s location
- Action: N,E,S,W
- State transition: move to the next grid according to the action
- Reward: -1 per time step

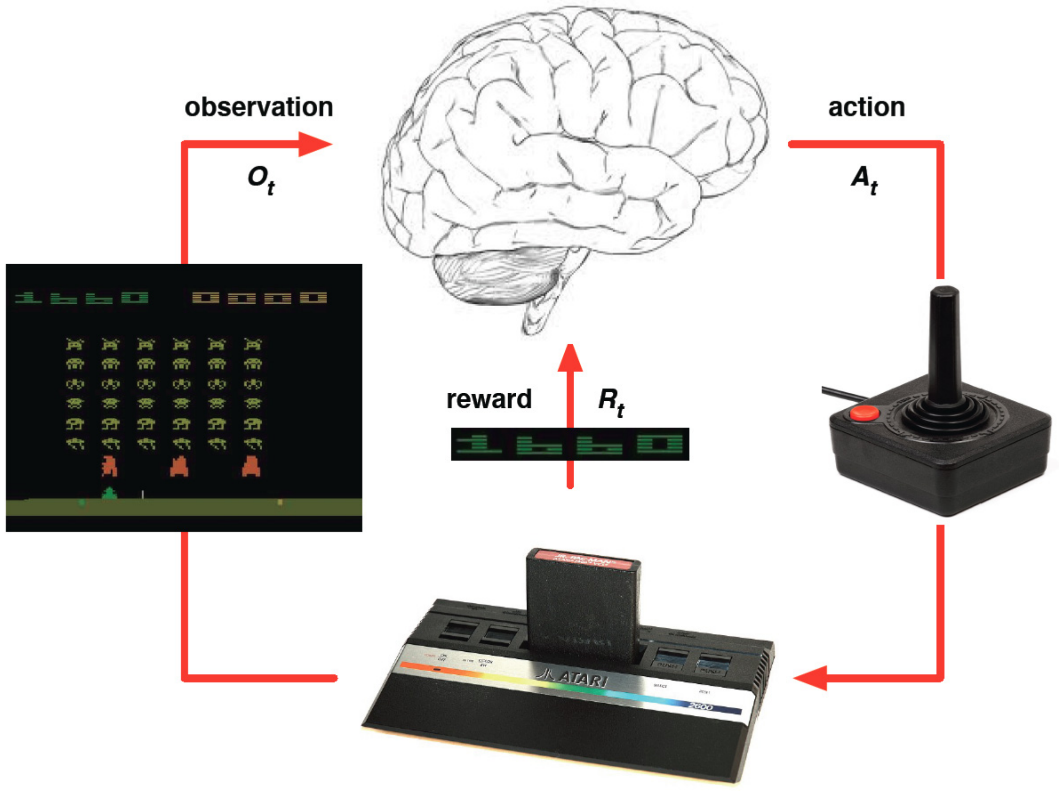
- Numbers represent value $v_{\pi}(s)$ of each state $s$

# Categorizing RL Agents

- Model based RL
  - Policy and/or value function
  - Model of the environment
  - E.g., the maze game above, game of Go

- Model-free RL
  - Policy and/or value function
  - No model of the environment
  - E.g., general playing Atari games

# Atari Example



- Rules of the game are unknown
- Learn from interactive game-play
- Pick actions on joystick, see pixels and scores

# Categorizing RL Agents

- Value based
  - No policy (implicit)
  - Value function

- Policy based
  - Policy
  - No value function

- Actor Critic
  - Policy
  - Value function

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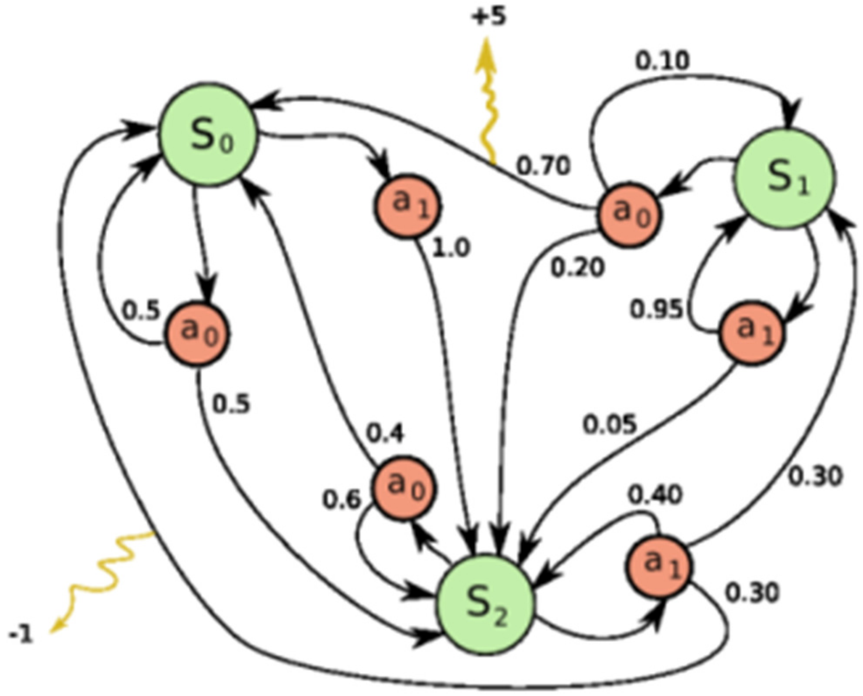
# Markov Decision Process

- Markov decision processes (MDPs) provide a mathematical framework
  - for modeling decision making in situations
  - where outcomes are partly random and partly under the control of a decision maker.



## Markov Decision Process

- Markov decision processes (MDPs) provide a mathematical framework
  - for modeling decision making in situations
  - where outcomes are partly random and partly under the control of a decision maker.

- MDPs formally describe an environment for RL
  - where the environment is FULLY observable
  - i.e. the current state completely characterizes the process (Markov property)

## Markov Property

"The future is independent of the past given the present"

- Definition
  - A state $S_t$ is Markov if and only if

$$\mathbb{P}[S_{t+1}|S_t] = \mathbb{P}[S_{t+1}|S_1, \ldots, S_t]$$

- Properties
  - The state captures all relevant information from the history
  - Once the state is known, the history may be thrown away
  - i.e. the state is sufficient statistic of the future

# Markov Decision Process

- A Markov decision process is a tuple ($S$, $A$, $\{P_{sa}\}$, $\gamma$, $R$)
- $S$ is the set of states
  - E.g., location in a maze, or current screen in an Atari game
- $A$ is the set of actions
  - E.g., move N, E, S, W, or the direction of the joystick and the buttons
- $P_{sa}$ are the state transition probabilities
  - For each state $s \in S$ and action $a \in A$, $P_{sa}$ is a distribution over the next state in $S$
- $\gamma \in [0,1]$ is the discount factor for the future reward
- $R : S \times A \mapsto \mathbb{R}$ is the reward function
  - Sometimes the reward is only assigned to state

# Markov Decision Process

The dynamics of an MDP proceeds as

- Start in a state $s_0$
- The agent chooses some action $a_0 \in A$
- The agent gets the reward $R(s_0, a_0)$
- MDP randomly transits to some successor state $s_1 \sim P_{s_0 a_0}$
- This proceeds iteratively

$$s_0 \xrightarrow[R(s_0,a_0)]{a_0} s_1 \xrightarrow[R(s_1,a_1)]{a_1} s_2 \xrightarrow[R(s_2,a_2)]{a_2} s_3 \cdots$$

- Until a terminal state $s_T$ or proceeds with no end
- The total payoff of the agent is

$$R(s_0, a_0) + \gamma R(s_1, a_1) + \gamma^2 R(s_2, a_2) + \cdots$$

# Reward on State Only

- For a large part of cases, reward is only assigned to the state
  - E.g., in maze game, the reward is on the location
  - In game of Go, the reward is only based on the final territory
- The reward function $R(s) : S \mapsto \mathbb{R}$
- MDPs proceed

$$s_0 \xrightarrow[R(s_0)]{a_0} s_1 \xrightarrow[R(s_1)]{a_1} s_2 \xrightarrow[R(s_2)]{a_2} s_3 \cdots$$

- cumulative reward (total payoff)

$$R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \cdots$$

# MDP Goal and Policy

- The goal is to choose actions over time to maximize the expected cumulative reward

$$\mathbb{E}[R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \cdots]$$

- $\gamma \in [0,1]$ is the discount factor for the future reward, which makes the agent prefer immediate reward to future reward
    - In finance case, today's \$1 is more valuable than \$1 in tomorrow
- Given a particular policy $\pi(s) : S \mapsto A$
    - i.e. take the action $a = \pi(s)$ at state *s*
- Define the value function for $\pi$

$$V^\pi(s) = \mathbb{E}[R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \cdots | s_0 = s, \pi]$$

- i.e. expected cumulative reward given the start state and taking actions according to $\pi$

# Bellman Equation for Value Function

- Define the value function for $\pi$

$$V^\pi(s) = \mathbb{E}[R(s_0) + \underbrace{\gamma R(s_1) + \gamma^2 R(s_2) + \cdots}_{\gamma V^\pi(s_1)} | s_0 = s, \pi]$$

$$= R(s) + \gamma \sum_{s' \in S} P_{s\pi(s)}(s') V^\pi(s')$$

Bellman Equation

Immediate Reward

Time decay

State transition

Value of the next state

# Optimal Value Function

- The optimal value function for each state *s* is best possible sum of discounted rewards that can be attained by any policy

$$V^*(s) = \max_\pi V^\pi(s)$$

- The Bellman's equation for optimal value function

$$V^*(s) = R(s) + \max_{a \in A} \gamma \sum_{s' \in S} P_{sa}(s') V^*(s')$$

- The optimal policy

$$\pi^*(s) = \arg\max_{a \in A} \sum_{s' \in S} P_{sa}(s') V^*(s')$$

- For every state *s* and every policy $\pi$

$$V^*(s) = V^{\pi^*}(s) \geq V^\pi(s)$$

# Value Iteration & Policy Iteration

- Note that the value function and policy are correlated

$$V^{\pi}(s) = R(s) + \gamma \sum_{s' \in S} P_{s\pi(s)}(s') V^{\pi}(s')$$

$$\pi(s) = \arg\max_{a \in A} \sum_{s' \in S} P_{sa}(s') V^{\pi}(s')$$

- It is feasible to perform iterative update towards the optimal value function and optimal policy
  - Value iteration
  - Policy iteration

# Value Iteration

- For an MDP with finite state and action spaces

$$|S| < \infty, |A| < \infty$$

- Value iteration is performed as

```
1. For each state s, initialize V(s) = 0.
2. Repeat until convergence {
        For each state, update
            $V(s) = R(s) + \max_{a \in A} \gamma \sum_{s' \in S} P_{sa}(s') V(s')$
   }
```

- Note that there is no explicit policy in above calculation

# Synchronous vs. Asynchronous VI

- Synchronous value iteration stores two copies of value functions

  1. For all *s* in *S*

  $$V_{\text{new}}(s) \leftarrow \max_{a \in A} \left( R(s) + \gamma \sum_{s' \in S} P_{sa}(s') V_{\text{old}}(s') \right)$$

  2. Update $V_{\text{old}}(s') \leftarrow V_{\text{new}}(s)$

- In-place asynchronous value iteration stores one copy of value function

  1. For all *s* in *S*

  $$V(s) \leftarrow \max_{a \in A} \left( R(s) + \gamma \sum_{s' \in S} P_{sa}(s') V(s') \right)$$

## Value Iteration Example: Shortest Path

| | | | |
|---|---|---|---|
| g | | | |
| | | | |
| | | | |
| | | | |

Problem

| | | | |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |

$V_1$

| | | | |
|---|---|---|---|
| 0 | -1 | -1 | -1 |
| -1 | -1 | -1 | -1 |
| -1 | -1 | -1 | -1 |
| -1 | -1 | -1 | -1 |

$V_2$

| | | | |
|---|---|---|---|
| 0 | -1 | -2 | -2 |
| -1 | -2 | -2 | -2 |
| -2 | -2 | -2 | -2 |
| -2 | -2 | -2 | -2 |

$V_3$

| | | | |
|---|---|---|---|
| 0 | -1 | -2 | -3 |
| -1 | -2 | -3 | -3 |
| -2 | -3 | -3 | -3 |
| -3 | -3 | -3 | -3 |

$V_4$

| | | | |
|---|---|---|---|
| 0 | -1 | -2 | -3 |
| -1 | -2 | -3 | -4 |
| -2 | -3 | -4 | -4 |
| -3 | -4 | -4 | -4 |

$V_5$

| | | | |
|---|---|---|---|
| 0 | -1 | -2 | -3 |
| -1 | -2 | -3 | -4 |
| -2 | -3 | -4 | -5 |
| -3 | -4 | -5 | -5 |

$V_6$

| | | | |
|---|---|---|---|
| 0 | -1 | -2 | -3 |
| -1 | -2 | -3 | -4 |
| -2 | -3 | -4 | -5 |
| -3 | -4 | -5 | -6 |

$V_7$

# Policy Iteration

- For an MDP with finite state and action spaces

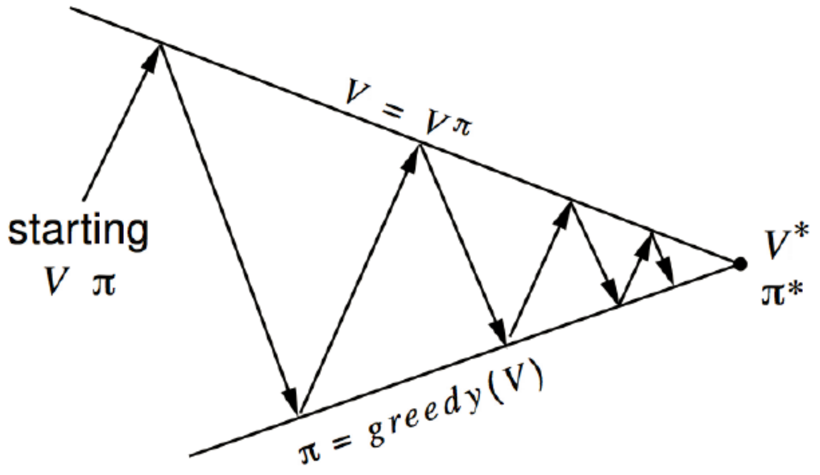
$$|S| < \infty, |A| < \infty$$

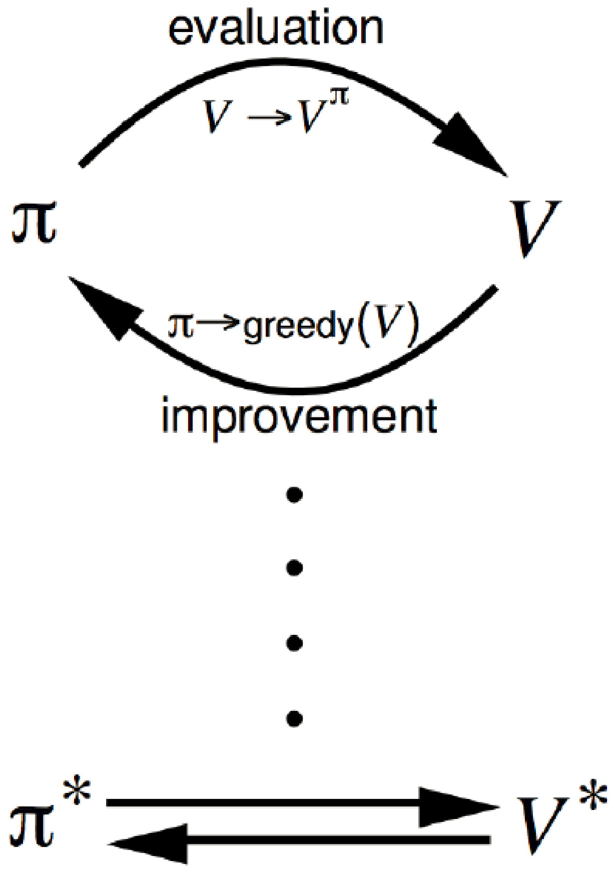
- Policy iteration is performed as

```
1. Initialize $\pi$ randomly
2. Repeat until convergence {
    a) Let $V := V^\pi$
    b) For each state, update
        $\pi(s) = \arg\max_{a \in A} \sum_{s' \in S} P_{sa}(s') V(s')$
   }
```

- The step of value function update could be time-consuming

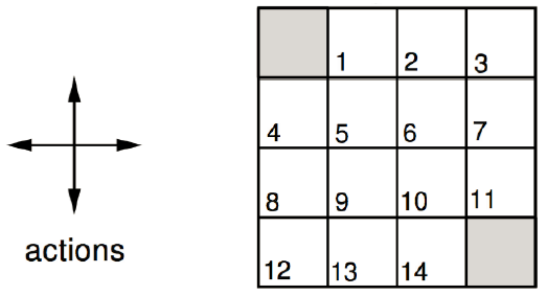
# Policy Iteration





- Policy evaluation
  - Estimate $V^\pi$
  - Iterative policy evaluation
- Policy improvement
  - Generate $\pi' \geq \pi$
  - Greedy policy improvement

# Evaluating a Random Policy in a Small Gridworld



*r* = -1
on all transitions

- Undiscounted episodic MDP (*γ*=1)
- Nonterminal states 1,…,14
- Two terminal states (shaded squares)
- Actions leading out of the grid leave state unchanged
- Reward is -1 until the terminal state is reached
- Agent follows a uniform random policy

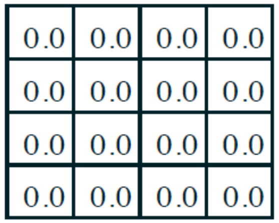
$$\pi(n|\cdot) = \pi(e|\cdot) = \pi(s|\cdot) = \pi(w|\cdot) = 0.25$$

# Evaluating a Random Policy in a Small Gridworld

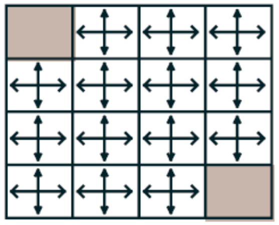
$V_k$ for the random policy

Greedy policy w.r.t. $V_k$

*K*=0

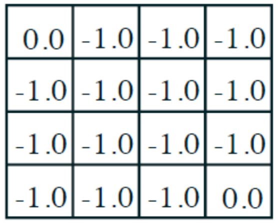


| | | | |
|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |

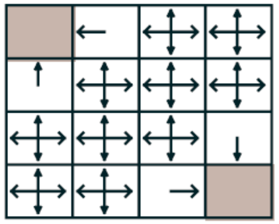


Random policy

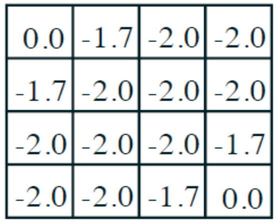
*K*=1



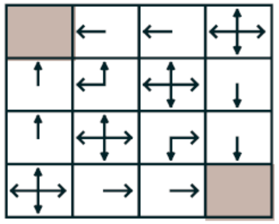
| | | | |
|---|---|---|---|
| 0.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | 0.0 |



*K*=2



| | | | |
|---|---|---|---|
| 0.0 | -1.7 | -2.0 | -2.0 |
| -1.7 | -2.0 | -2.0 | -2.0 |
| -2.0 | -2.0 | -2.0 | -1.7 |
| -2.0 | -2.0 | -1.7 | 0.0 |



# Evaluating a Random Policy in a Small Gridworld

$V_k$ for the random policy

Greedy policy w.r.t. $V_k$

*K*=3

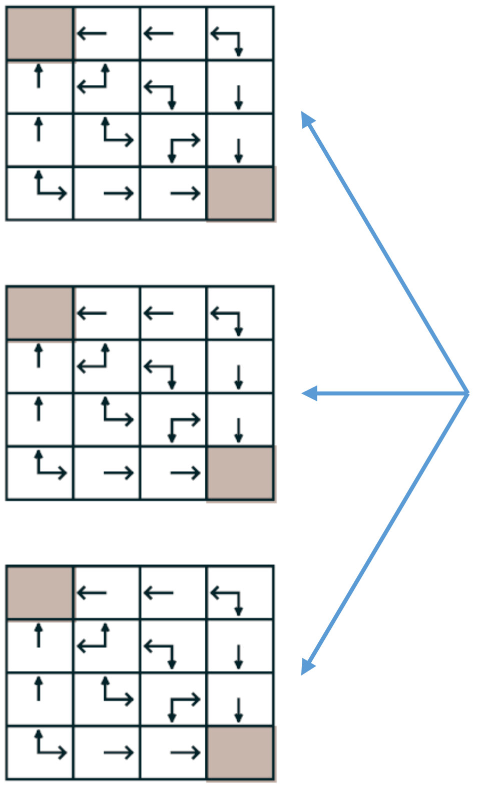
| | | | |
|---|---|---|---|
| 0.0 | -2.4 | -2.9 | -3.0 |
| -2.4 | -2.9 | -3.0 | -2.9 |
| -2.9 | -3.0 | -2.9 | -2.4 |
| -3.0 | -2.9 | -2.4 | 0.0 |

*K*=10

| | | | |
|---|---|---|---|
| 0.0 | -6.1 | -8.4 | -9.0 |
| -6.1 | -7.7 | -8.4 | -8.4 |
| -8.4 | -8.4 | -7.7 | -6.1 |
| -9.0 | -8.4 | -6.1 | 0.0 |

*K*=∞

| | | | |
|---|---|---|---|
| 0.0 | -14. | -20. | -22. |
| -14. | -18. | -20. | -20. |
| -20. | -20. | -18. | -14. |
| -22. | -20. | -14. | 0.0 |



$V := V^{\pi}$

Optimal policy

# Value Iteration vs. Policy Iteration

## Value iteration

1. For each state *s*, initialize *V*(*s*) = 0.
2. Repeat until convergence {

   For each state, update

$$V(s) = R(s) + \max_{a \in A} \gamma \sum_{s' \in S} P_{sa}(s') V(s')$$

}

## Policy iteration

1. Initialize *π* randomly
2. Repeat until convergence {
   a) Let $V := V^{\pi}$
   b) For each state, update

$$\pi(s) = \arg\max_{a \in A} \sum_{s' \in S} P_{sa}(s') V(s')$$

}

Remarks:

1. Value iteration is a greedy update strategy
2. In policy iteration, the value function update by bellman equation is costly
3. For small-space MDPs, policy iteration is often very fast and converges quickly
4. For large-space MDPs, value iteration is more practical (efficient)
5. If there is no state-transition loop, it is better to use value iteration

My point of view: value iteration is like SGD and policy iteration is like BGD

# Learning an MDP Model

- So far we have been focused on
  - Calculating the optimal value function
  - Learning the optimal policy

  given a known MDP model
  - i.e. the state transition $P_{sa}(s')$ and reward function $R(s)$ are explicitly given
- In realistic problems, often the state transition and reward function are not explicitly given
  - For example, we have only observed some episodes

Episode 1: $s_0^{(1)} \xrightarrow[R(s_0)^{(1)}]{a_0^{(1)}} s_1^{(1)} \xrightarrow[R(s_1)^{(1)}]{a_1^{(1)}} s_2^{(1)} \xrightarrow[R(s_2)^{(1)}]{a_2^{(1)}} s_3^{(1)} \cdots s_T^{(1)}$

Episode 2: $s_0^{(2)} \xrightarrow[R(s_0)^{(2)}]{a_0^{(2)}} s_1^{(2)} \xrightarrow[R(s_1)^{(2)}]{a_1^{(2)}} s_2^{(2)} \xrightarrow[R(s_2)^{(2)}]{a_2^{(2)}} s_3^{(2)} \cdots s_T^{(2)}$

# Learning an MDP Model

Episode 1: $s_0^{(1)} \xrightarrow[R(s_0)^{(1)}]{a_0^{(1)}} s_1^{(1)} \xrightarrow[R(s_1)^{(1)}]{a_1^{(1)}} s_2^{(1)} \xrightarrow[R(s_2)^{(1)}]{a_2^{(1)}} s_3^{(1)} \cdots s_T^{(1)}$

Episode 2: $s_0^{(2)} \xrightarrow[R(s_0)^{(2)}]{a_0^{(2)}} s_1^{(2)} \xrightarrow[R(s_1)^{(2)}]{a_1^{(2)}} s_2^{(2)} \xrightarrow[R(s_2)^{(2)}]{a_2^{(2)}} s_3^{(2)} \cdots s_T^{(2)}$

$\vdots$ $\qquad\qquad\qquad\qquad$ $\vdots$

- Learn an MDP model from "experience"
  - Learning state transition probabilities *P*$_{sa}$(*s'*)

$$P_{sa}(s') = \frac{\#\text{times we took action } a \text{ in state } s \text{ and got to state } s'}{\#\text{times we took action } a \text{ in state } s}$$

  - Learning reward *R*(*s*), i.e. the expected immediate reward

$$R(s) = \text{average}\Big\{R(s)^{(i)}\Big\}$$

## Learning Model and Optimizing Policy

- Algorithm

```
1. Initialize $\pi$ randomly.
2. Repeat until convergence {
    a) Execute $\pi$ in the MDP for some number of trials
    b) Using the accumulated experience in the MDP, update our
       estimates for $P_{sa}$ and $R$
    c) Apply value iteration with the estimated $P_{sa}$ and $R$ to get the
       new estimated value function $V$
    d) Update $\pi$ to be the greedy policy w.r.t. $V$
   }
```

# Learning an MDP Model

- In realistic problems, often the state transition and reward function are not explicitly given
  - For example, we have only observed some episodes

Episode 1: $s_0^{(1)} \xrightarrow[R(s_0)^{(1)}]{a_0^{(1)}} s_1^{(1)} \xrightarrow[R(s_1)^{(1)}]{a_1^{(1)}} s_2^{(1)} \xrightarrow[R(s_2)^{(1)}]{a_2^{(1)}} s_3^{(1)} \cdots s_T^{(1)}$

Episode 2: $s_0^{(2)} \xrightarrow[R(s_0)^{(2)}]{a_0^{(2)}} s_1^{(2)} \xrightarrow[R(s_1)^{(2)}]{a_1^{(2)}} s_2^{(2)} \xrightarrow[R(s_2)^{(2)}]{a_2^{(2)}} s_3^{(2)} \cdots s_T^{(2)}$

- Another branch of solution is to directly learning value & policy from experience without building an MDP
- i.e. Model-free Reinforcement Learning

## Content

- Introduction to Reinforcement Learning

- Model-based Reinforcement Learning
  - Markov Decision Process
  - Planning by Dynamic Programming

- Model-free Reinforcement Learning
  - Model-free Prediction
    - Monte-Carlo and Temporal Difference
  - Model-free Control
    - On-policy SARSA and off-policy Q-learning

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# Model-free Reinforcement Learning

- In realistic problems, often the state transition and reward function are not explicitly given
  - For example, we have only observed some episodes

Episode 1: $s_0^{(1)} \xrightarrow[R(s_0)^{(1)}]{a_0^{(1)}} s_1^{(1)} \xrightarrow[R(s_1)^{(1)}]{a_1^{(1)}} s_2^{(1)} \xrightarrow[R(s_2)^{(1)}]{a_2^{(1)}} s_3^{(1)} \cdots s_T^{(1)}$

Episode 2: $s_0^{(2)} \xrightarrow[R(s_0)^{(2)}]{a_0^{(2)}} s_1^{(2)} \xrightarrow[R(s_1)^{(2)}]{a_1^{(2)}} s_2^{(2)} \xrightarrow[R(s_2)^{(2)}]{a_2^{(2)}} s_3^{(2)} \cdots s_T^{(2)}$

- Model-free RL is to directly learn value & policy from experience without building an MDP
- Key steps: (1) estimate value function; (2) optimize policy

## Value Function Estimation

- In model-based RL (MDP), the value function is calculated by dynamic programming

$$\begin{aligned} V^\pi(s) &= \mathbb{E}[R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \cdots | s_0 = s, \pi] \\ &= R(s) + \gamma \sum_{s' \in S} P_{s\pi(s)}(s') V^\pi(s') \end{aligned}$$

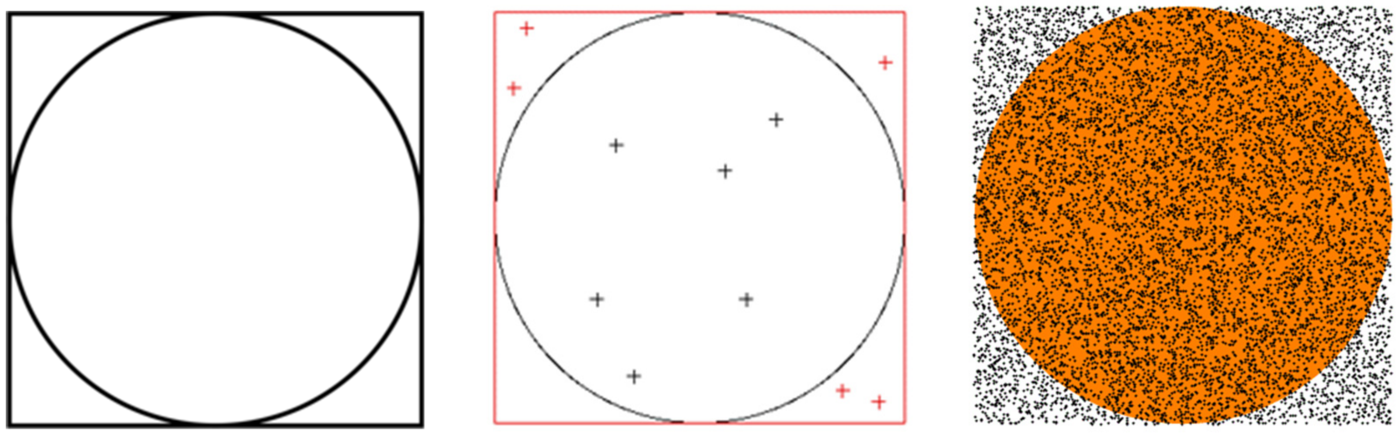
- Now in model-free RL
  - We cannot directly know *P*<sub></sub>
  - But we have a list of experiences to estimate the values

Episode 1: $s_0^{(1)} \xrightarrow[R(s_0)^{(1)}]{a_0^{(1)}} s_1^{(1)} \xrightarrow[R(s_1)^{(1)}]{a_1^{(1)}} s_2^{(1)} \xrightarrow[R(s_2)^{(1)}]{a_2^{(1)}} s_3^{(1)} \cdots s_T^{(1)}$

Episode 2: $s_0^{(2)} \xrightarrow[R(s_0)^{(2)}]{a_0^{(2)}} s_1^{(2)} \xrightarrow[R(s_1)^{(2)}]{a_1^{(2)}} s_2^{(2)} \xrightarrow[R(s_2)^{(2)}]{a_2^{(2)}} s_3^{(2)} \cdots s_T^{(2)}$

## Monte-Carlo Methods

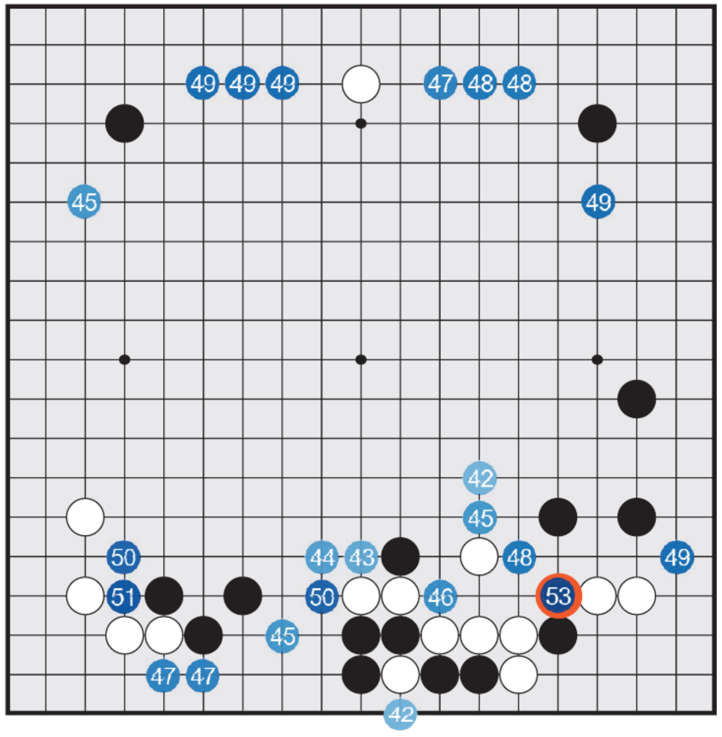
- Monte-Carlo methods are a broad class of computational algorithms that rely on repeated random sampling to obtain numerical results.
- Example, to calculate the circle's surface

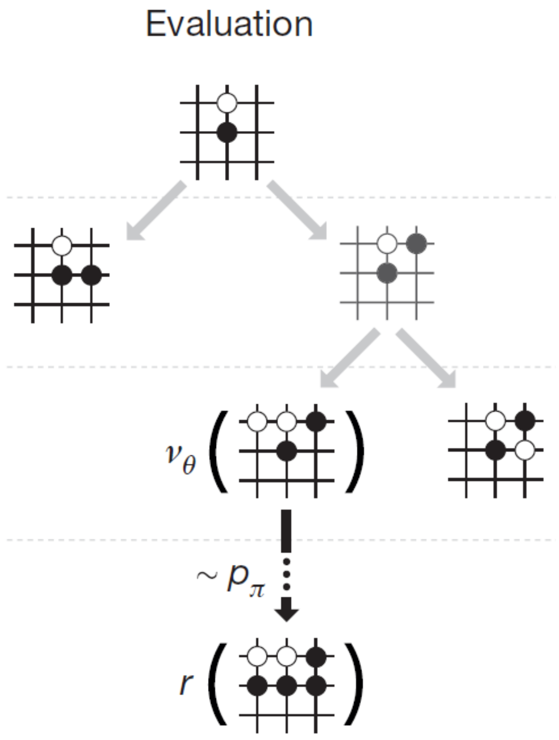


$$\text{Circle Surface} = \text{Square Surface} \times \frac{\#\text{points in circle}}{\#\text{points in total}}$$

# Monte-Carlo Methods

- Go: to estimate the winning rate given the current state





$$\text{Win Rate}(s) = \frac{\#\text{win simulation cases started from } s}{\#\text{simulation cases started from } s \text{ in total}}$$

# Monte-Carlo Value Estimation

- Goal: learn $V^\pi$ from episodes of experience under policy $\pi$

$$s_0^{(i)} \xrightarrow[R_1^{(i)}]{a_0^{(i)}} s_1^{(i)} \xrightarrow[R_2^{(i)}]{a_1^{(i)}} s_2^{(i)} \xrightarrow[R_3^{(i)}]{a_2^{(i)}} s_3^{(i)} \cdots s_T^{(i)} \sim \pi$$

- Recall that the return is the total discounted reward

$$G_t = R_{t+1} + \gamma R_{t+2} + \ldots \gamma^{T-1} R_T$$

- Recall that the value function is the expected return

$$\begin{aligned} V^\pi(s) &= \mathbb{E}[R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \cdots | s_0 = s, \pi] \\ &= \mathbb{E}[G_t | s_t = s, \pi] \\ &\simeq \frac{1}{N} \sum_{i=1}^{N} G_t^{(i)} \end{aligned}$$

- Sample *N* episodes from state *s* using policy *π*
- Calculate the average of cumulative reward

- Monte-Carlo policy evaluation uses empirical mean return instead of expected return

# Monte-Carlo Value Estimation

Idea: $$V(S_t) \simeq \frac{1}{N}\sum_{i=1}^{N} G_t^{(i)}$$

Implementation: $$V(S_t) \leftarrow V(S_t) + \alpha(G_t - V(S_t))$$

- MC methods learn directly from episodes of experience
- MC is model-free: no knowledge of MDP transitions / rewards
- MC learns from **complete** episodes: no bootstrapping (discussed later)
- MC uses the simplest possible idea: value = mean return
- Caveat: can only apply MC to episodic MDPs
  - All episodes must terminate

# Temporal-Difference Learning

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots = R_{t+1} + \gamma V(S_{t+1})$$

$$V(S_t) \leftarrow V(S_t) + \alpha(\underset{\substack{\uparrow \\ \text{Observation}}}{R_{t+1}} + \gamma V(\underset{\substack{\uparrow \\ \text{Guess of} \\ \text{future}}}{S_{t+1}}) - V(S_t))$$

- TD methods learn directly from episodes of experience
- TD is model-free: no knowledge of MDP transitions / rewards
- TD learns from incomplete episodes, by bootstrapping
- TD updates a guess towards a guess

# Monte Carlo vs. Temporal Difference

- The same goal: learn $V^{\pi}$ from episodes of experience under policy $\pi$
- Incremental every-visit Monte-Carlo
  - Update value $V(S_t)$ toward actual return $G_t$

$$V(S_t) \leftarrow V(S_t) + \alpha(G_t - V(S_t))$$

- Simplest temporal-difference learning algorithm: TD
  - Update value $V(S_t)$ toward estimated return $R_{t+1} + \gamma V(S_{t+1})$

$$V(S_t) \leftarrow V(S_t) + \alpha(R_{t+1} + \gamma V(S_{t+1}) - V(S_t))$$

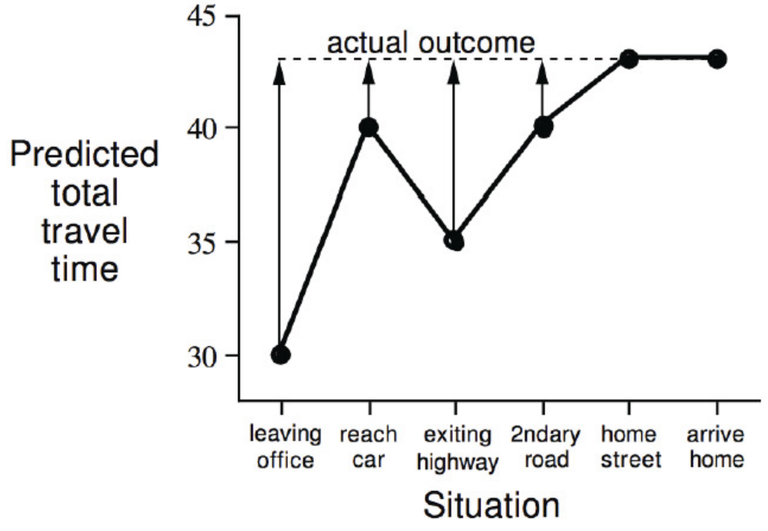
  - TD target: $R_{t+1} + \gamma V(S_{t+1})$
  - TD error: $\delta_t = R_{t+1} + \gamma V(S_{t+1}) - V(S_t)$

# Driving Home Example

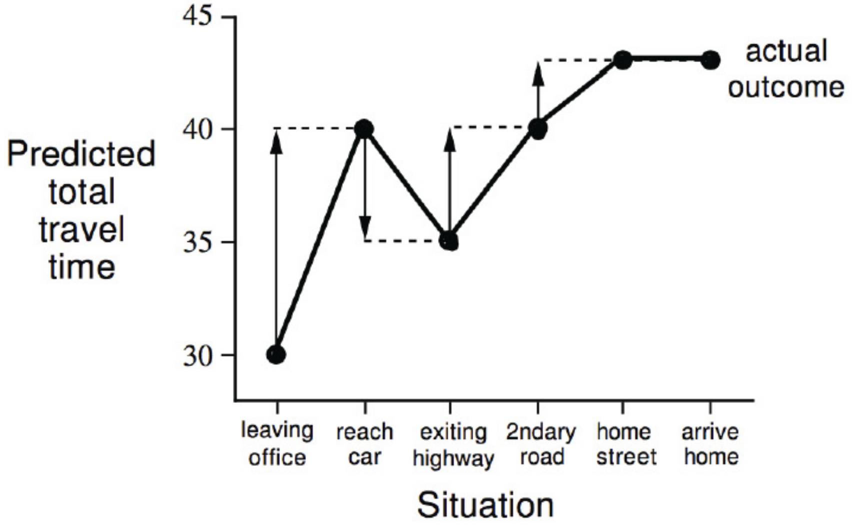
| State | Elapsed Time (Minutes) | Predicted Time to Go | Predicted Total Time |
|---|---|---|---|
| Leaving office | 0 | 30 | 30 |
| Reach car, raining | 5 | 35 | 40 |
| Exit highway | 20 | 15 | 35 |
| Behind truck | 30 | 10 | 40 |
| Home street | 40 | 3 | 43 |
| Arrow home | 43 | 0 | 43 |

# Driving Home Example: MC vs. TD

Changes recommended by Monte Carlo methods (α=1)



Changes recommended by TD methods (α=1)



## Advantages and Disadvantages of MC vs. TD

- TD can learn before knowing the final outcome
  - TD can learn online after every step
  - MC must wait until end of episode before return is known

- TD can learn without the final outcome
  - TD can learn from incomplete sequences
  - MC can only learn from complete sequences
  - TD works in continuing (non-terminating) environments
  - MC only works for episodic (terminating) environments

# Bias/Variance Trade-Off

- Return $G_t = R_{t+1} + \gamma R_{t+2} + \ldots + \gamma^{T-1} R_T$ is unbiased estimate of $V^\pi(S_t)$
- True TD target $R_{t+1} + \gamma V^\pi(S_{t+1})$ is unbiased estimate of $V^\pi(S_t)$
- TD target $R_{t+1} + \gamma \underbrace{V(S_{t+1})}_{\text{current estimate}}$ is biased estimate of $V^\pi(S_t)$

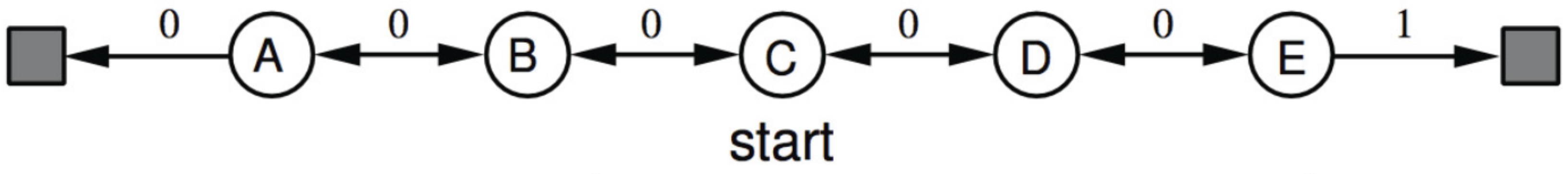
- TD target is of much lower variance than the return
  - Return depends on many random actions, transitions and rewards
  - TD target depends on one random action, transition and reward

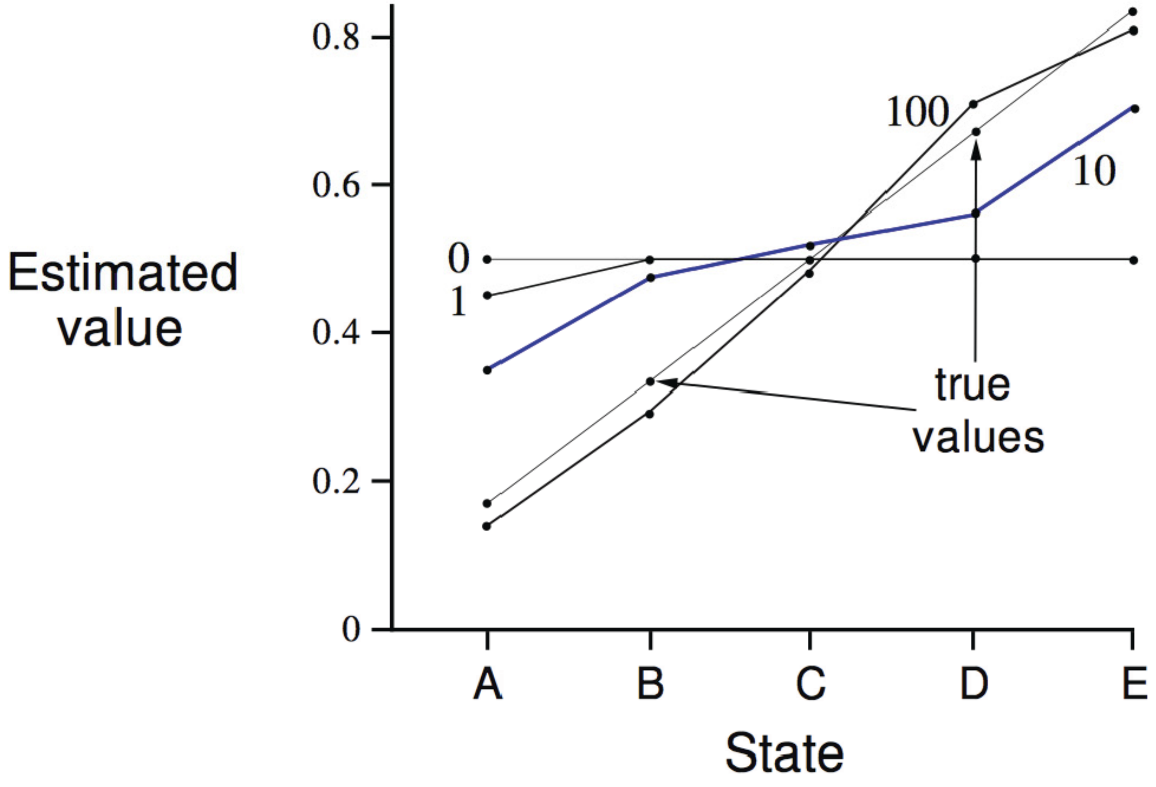
## Advantages and Disadvantages of MC vs. TD (2)

- MC has high variance, zero bias
  - Good convergence properties
  - (even with function approximation)
  - Not very sensitive to initial value
  - Very simple to understand and use

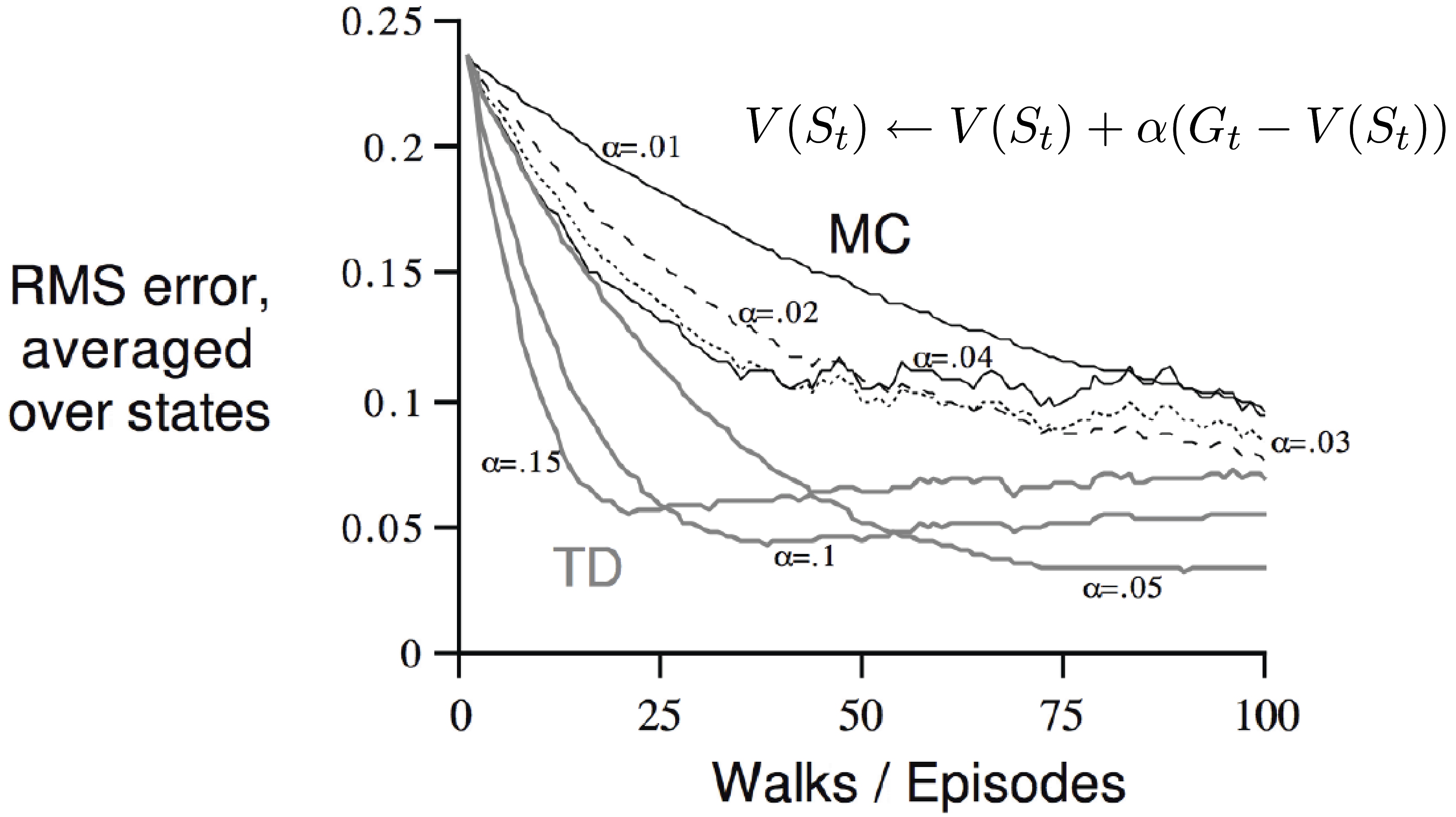
- TD has low variance, some bias
  - Usually more efficient than MC
  - TD converges to $V^{\pi}(S_t)$
    - (but not always with function approximation)
  - More sensitive to initial value than MC

# Random Walk Example





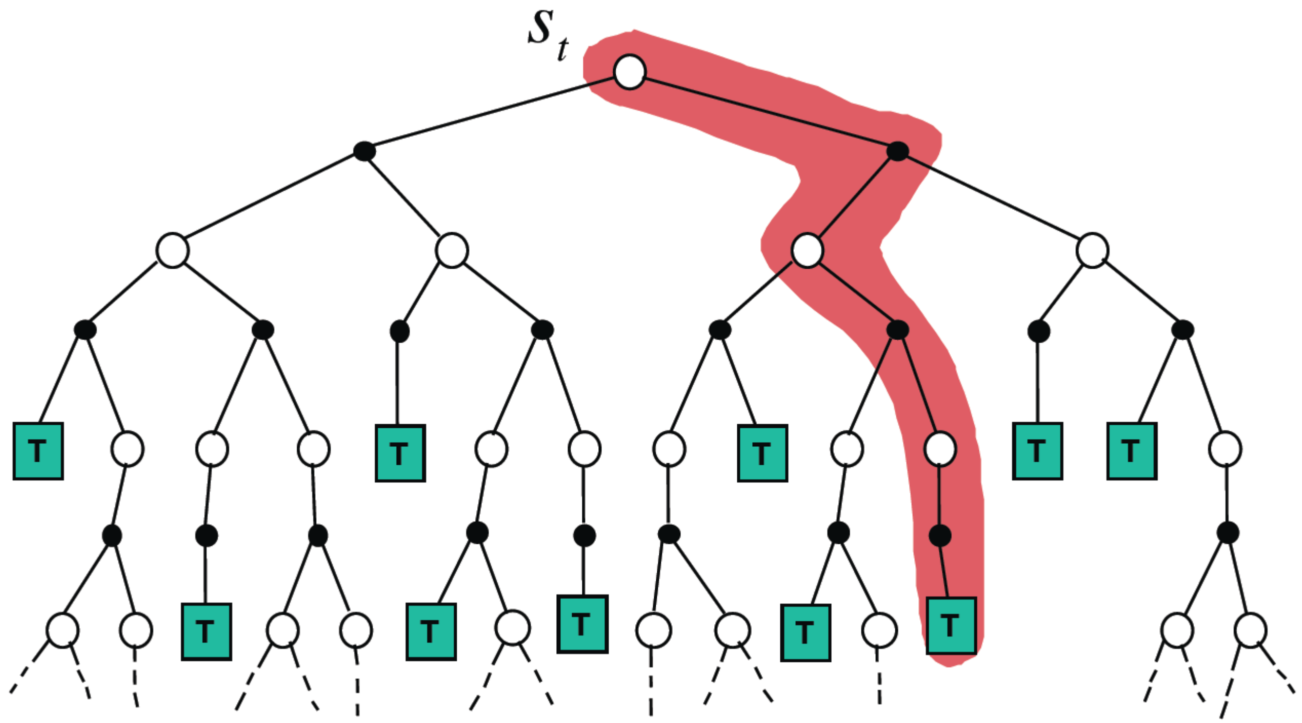
# Random Walk Example



$$V(S_t) \leftarrow V(S_t) + \alpha(R_{t+1} + \gamma V(S_{t+1}) - V(S_t))$$

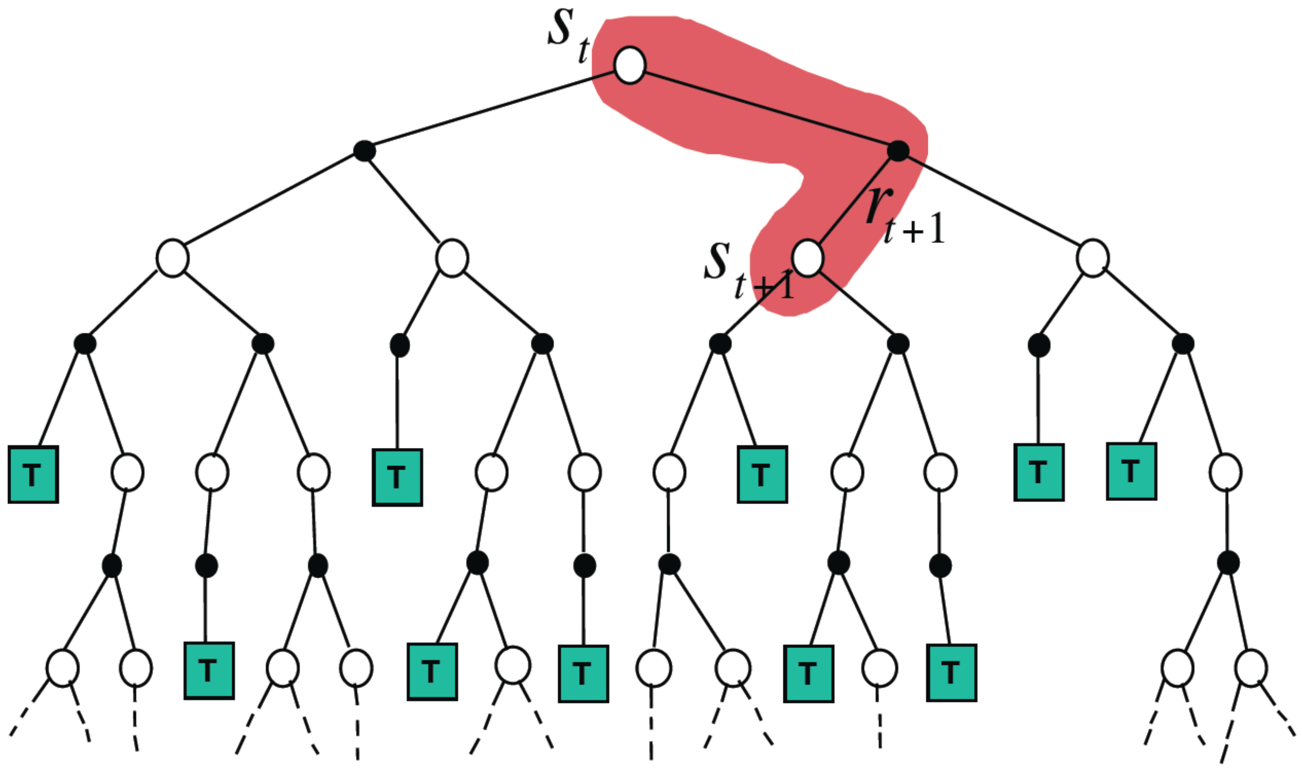
# Monte-Carlo Backup

$$V(S_t) \leftarrow V(S_t) + \alpha(G_t - V(S_t))$$



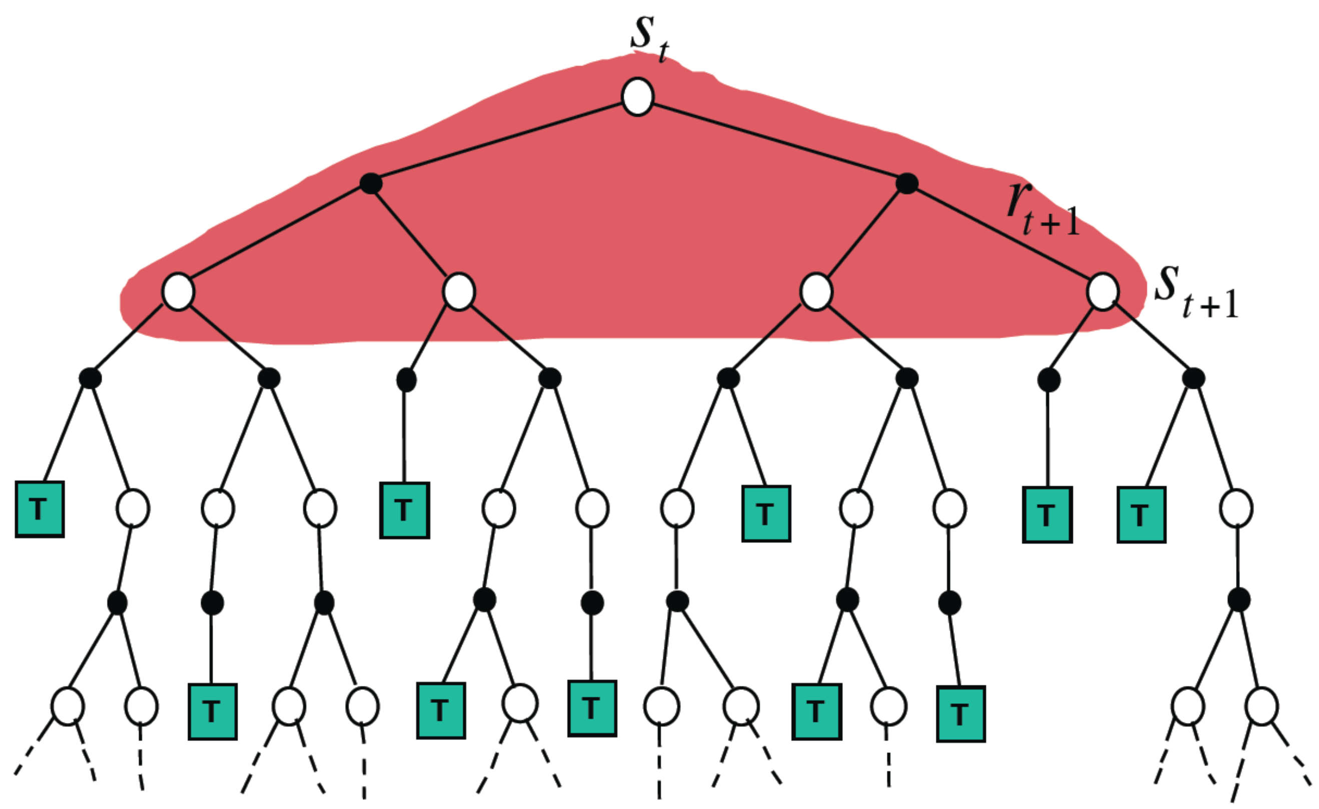
# Temporal-Difference Backup

$$V(S_t) \leftarrow V(S_t) + \alpha(R_{t+1} + \gamma V(S_{t+1}) - V(S_t))$$



# Dynamic Programming Backup

$$V(S_t) \leftarrow \mathbb{E}[R_{t+1} + \gamma V(S_{t+1})]$$



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## Uses of Model-Free Control

- Some example problems that can be modeled as MDPs
    - Elevator
    - Parallel parking
    - Ship steering
    - Bioreactor
    - Helicopter
    - Aeroplane logistics
    - Robocup soccer
    - Atari & StarCraft
    - Portfolio management
    - Protein folding
    - Robot walking
    - Game of Go
- For most of real-world problems, either:
    - MDP model is unknown, but experience can be sampled
    - MDP model is known, but is too big to use, except by samples
- Model-free control can solve these problems

# On- and Off-Policy Learning

- Two categories of model-free RL

- On-policy learning
    - "Learn on the job"
    - Learn about policy $\pi$ from experience sampled from $\pi$

- Off-policy learning
    - "Look over someone's shoulder"
    - Learn about policy $\pi$ from experience sampled from another policy $\mu$

# State Value and Action Value

$$G_t = R_{t+1} + \gamma R_{t+2} + \dots \gamma^{T-1} R_T$$

- State value
  - The state-value function $V^{\pi}(s)$ of an MDP is the expected return starting from state *s* and then following policy *π*

$$V^{\pi}(s) = \mathbb{E}_{\pi}[G_t | S_t = s]$$

- Action value
  - The action-value function $Q^{\pi}(s,a)$ of an MDP is the expected return starting from state *s*, taking action *a*, and then following policy *π*

$$Q^{\pi}(s,a) = \mathbb{E}_{\pi}[G_t | S_t = s, A_t = a]$$

# Bellman Expectation Equation

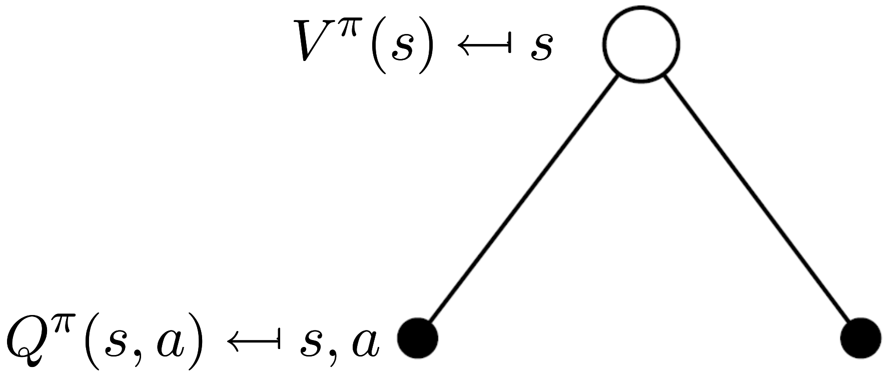
- The state-value function $V^{\pi}(s)$ can be decomposed into immediate reward plus discounted value of successor state

$$V^{\pi}(s) = \mathbb{E}_{\pi}[R_{t+1} + \gamma V^{\pi}(S_{t+1}) | S_t = s]$$

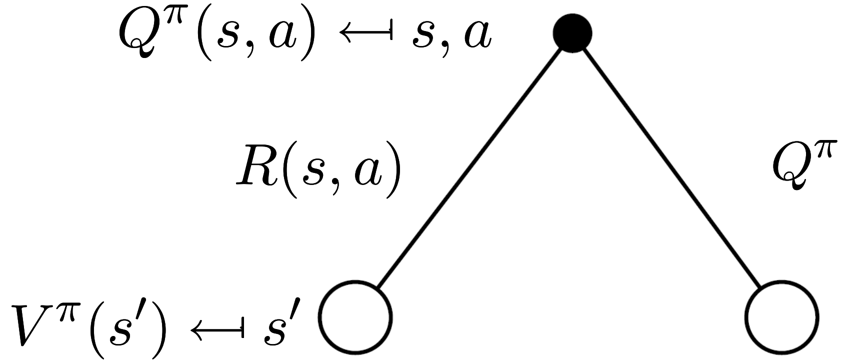
- The action-value function $Q^{\pi}(s,a)$ can similarly be decomposed

$$Q^{\pi}(s,a) = \mathbb{E}_{\pi}[R_{t+1} + \gamma Q^{\pi}(S_{t+1}, A_{t+1}) | S_t = s, A_t = a]$$

# State Value and Action Value



$$V^\pi(s) = \sum_{a \in A} \pi(a|s) Q^\pi(s,a)$$



$Q^\pi(s,a) \hookleftarrow s,a$
$R(s,a)$
$V^\pi(s') \hookleftarrow s'$

$$Q^\pi(s,a) = R(s,a) + \gamma \sum_{s' \in S} P_{sa}(s') V^\pi(s')$$

# Model-Free Policy Iteration

- Given state-value function *V*(*s*) and action-value function *Q*(*s,a*), model-free policy iteration shall use action-value function

- Greedy policy improvement over *V*(*s*) requires model of MDP

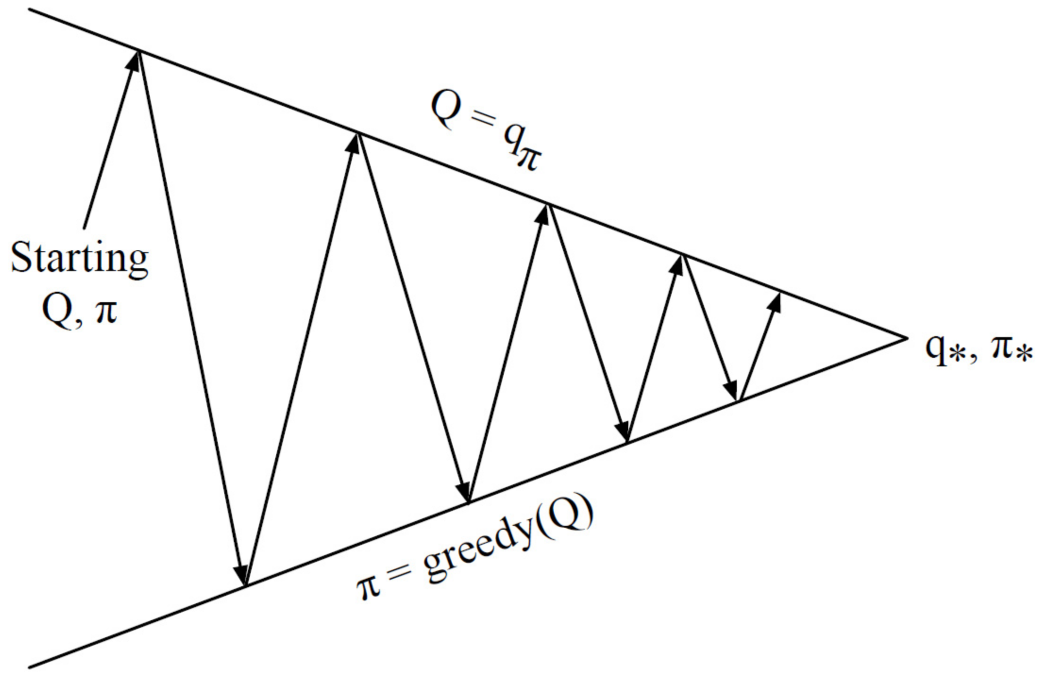
$$\pi^{\text{new}}(s) = \arg\max_{a \in A} \left\{ R(s,a) + \gamma \sum_{s' \in S} P_{sa}(s') V^{\pi}(s') \right\}$$

We don’t know the transition probability

- Greedy policy improvement over *Q*(*s,a*) is model-free

$$\pi^{\text{new}}(s) = \arg\max_{a \in A} Q(s,a)$$

## Generalized Policy Iteration with Action-Value Function



- Policy evaluation: Monte-Carlo policy evaluation, $Q = Q^\pi$
- Policy improvement: Greedy policy improvement?

# Example of Greedy Action Selection

- Greedy policy improvement over *Q*(*s*,*a*) is model-free

$$\pi^{\text{new}}(s) = \arg\max_{a \in A} Q(s,a)$$

- Given the right example
  - What if the first action is to choose the left door and observe reward=0?
  - The policy would be suboptimal if there is no exploration

| Left: | Right: |
|---|---|
| 20% Reward = 0 | 50% Reward = 1 |
| 80% Reward = 5 | 50% Reward = 3 |

"Behind one door is tenure – behind the other is flipping burgers at McDonald's."

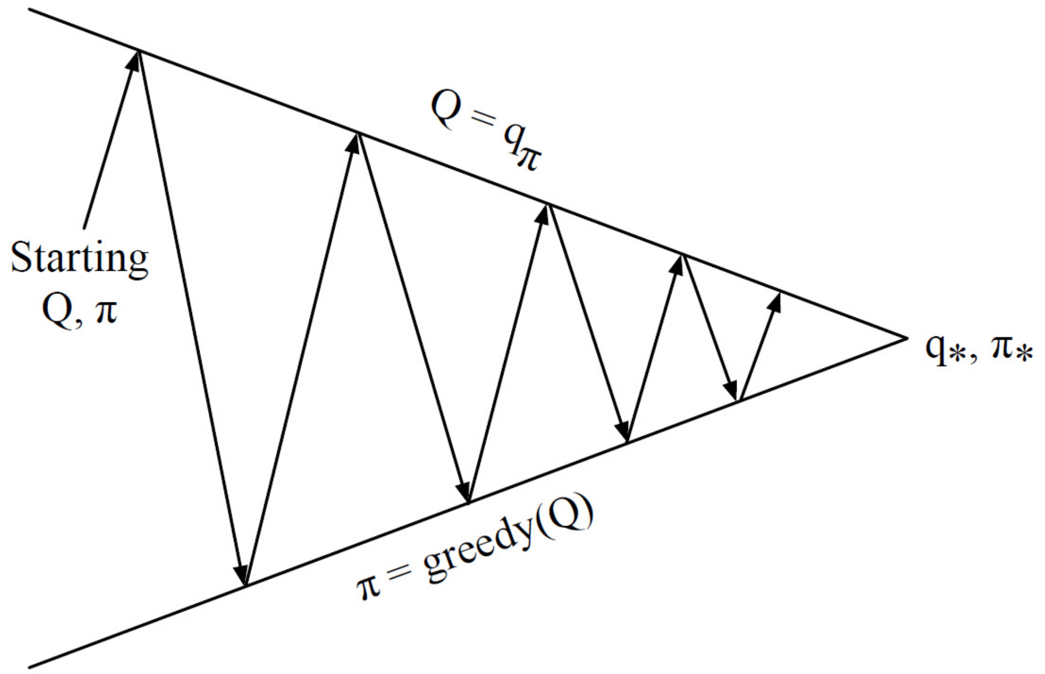
# *ε*-Greedy Policy Exploration

- Simplest idea for ensuring continual exploration
- All *m* actions are tried with non-zero probability
- With probability 1-*ε*, choose the greedy action
- With probability *ε*, choose an action at random

$$\pi(a|s) = \begin{cases} \epsilon/m + 1 - \epsilon & \text{if } a^* = \arg\max_{a \in A} Q(s,a) \\ \epsilon/m & \text{otherwise} \end{cases}$$

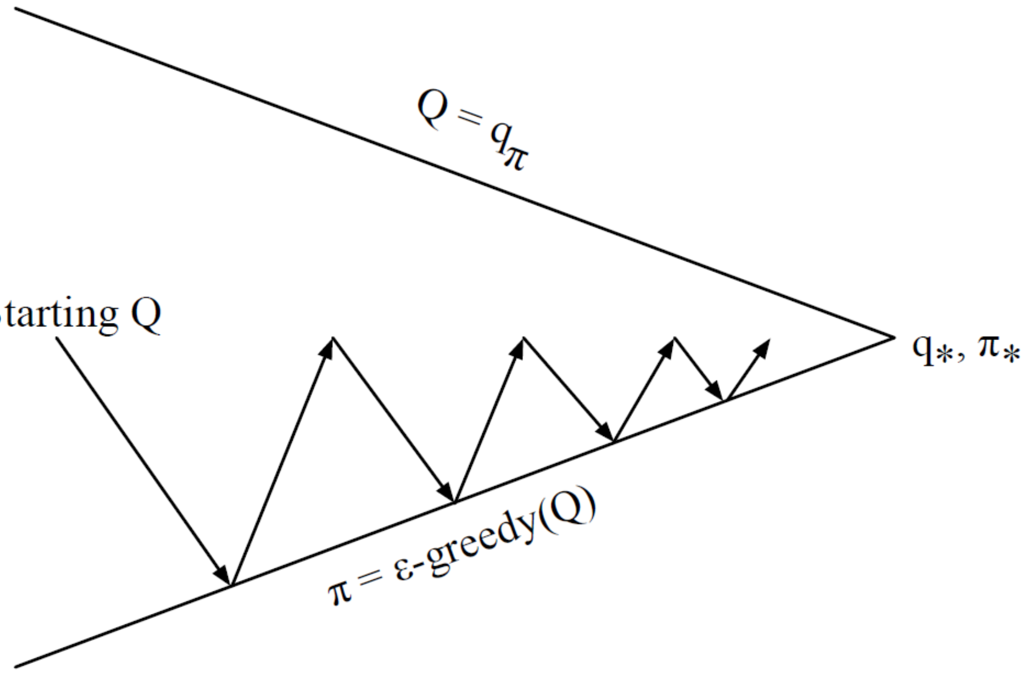
- Theorem
  - For any *ε*-greedy policy *π*, the *ε*-greedy policy *π*’ w.r.t. $Q^\pi$ is an improvement, i.e. $V^{\pi'}(s) \geq V^\pi(s)$

## Generalized Policy Iteration with Action-Value Function



- Policy evaluation: Monte-Carlo policy evaluation, $Q = Q^{\pi}$
- Policy improvement: $\varepsilon$-greedy policy improvement

## Monte-Carlo Control



Every episode:

- Policy evaluation: Monte-Carlo policy evaluation, $Q \approx Q^\pi$
- Policy improvement: $\varepsilon$-greedy policy improvement

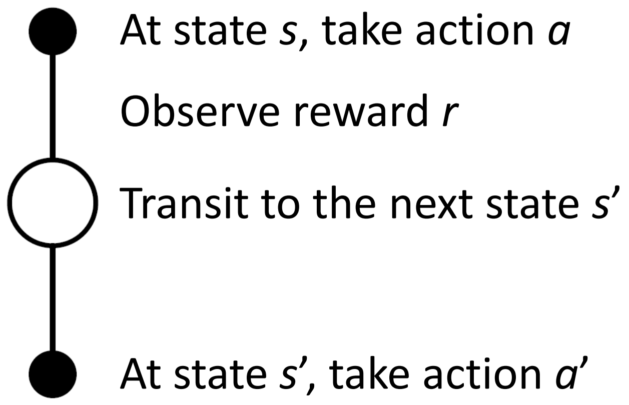
## MC Control vs. TD Control

- Temporal-difference (TD) learning has several advantages over Monte-Carlo (MC)
  - Lower variance
  - Online
  - Incomplete sequences

- Natural idea: use TD instead of MC in our control loop
  - Apply TD to update action value $Q(s,a)$
  - Use $\varepsilon$-greedy policy improvement
  - Update the action value function every time-step

# SARSA

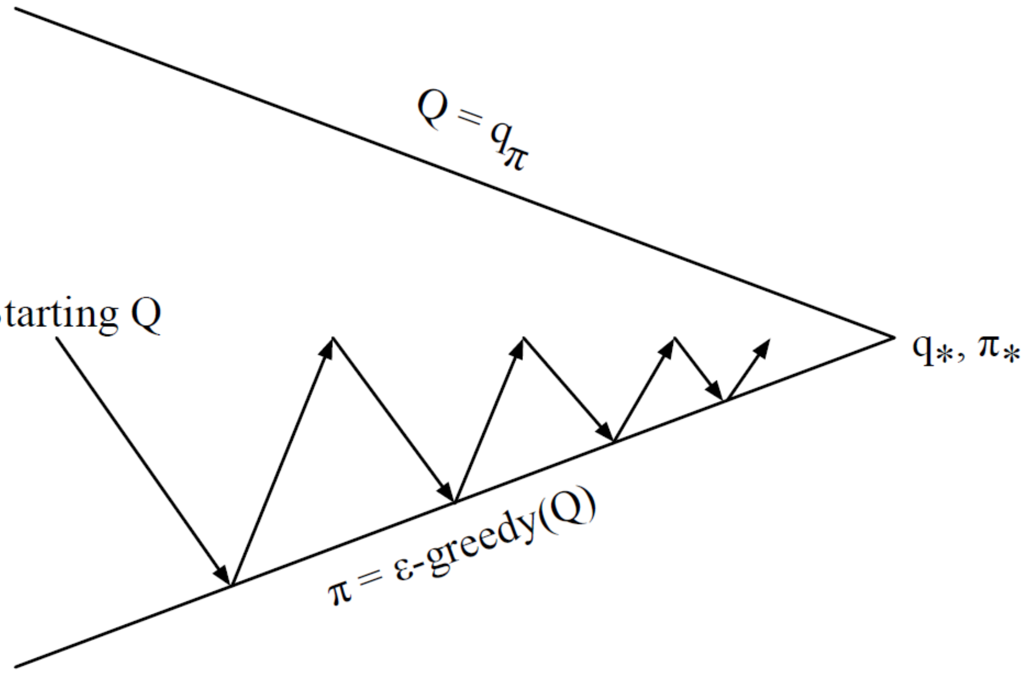
- For each state-action-reward-state-action by the current policy


At state *s*, take action *a*
Observe reward *r*
Transit to the next state *s’*
At state *s’*, take action *a’*

- Updating action-value functions with Sarsa

$$Q(s,a) \leftarrow Q(s,a) + \alpha(r + \gamma Q(s',a') - Q(s,a))$$

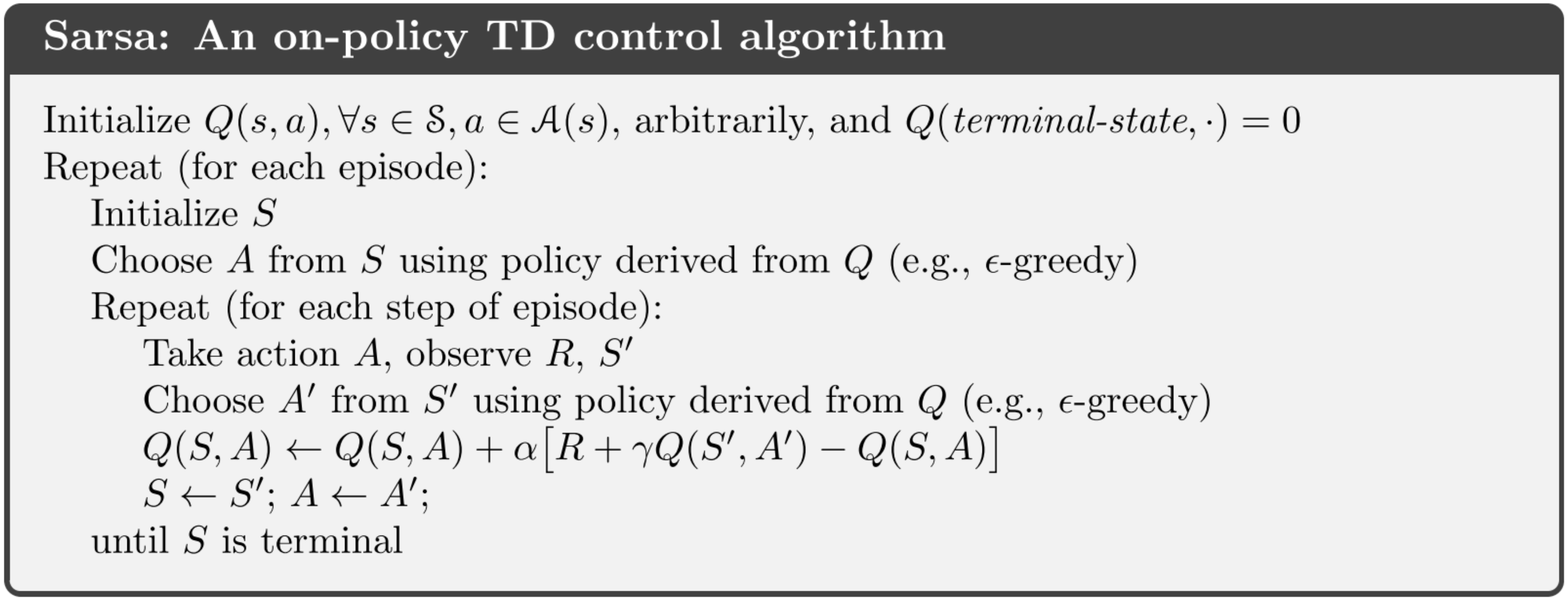
## On-Policy Control with SARSA



Every time-step:

- Policy evaluation: Sarsa $Q(s,a) \leftarrow Q(s,a) + \alpha(r + \gamma Q(s',a') - Q(s,a))$
- Policy improvement: $\varepsilon$-greedy policy improvement

# SARSA Algorithm

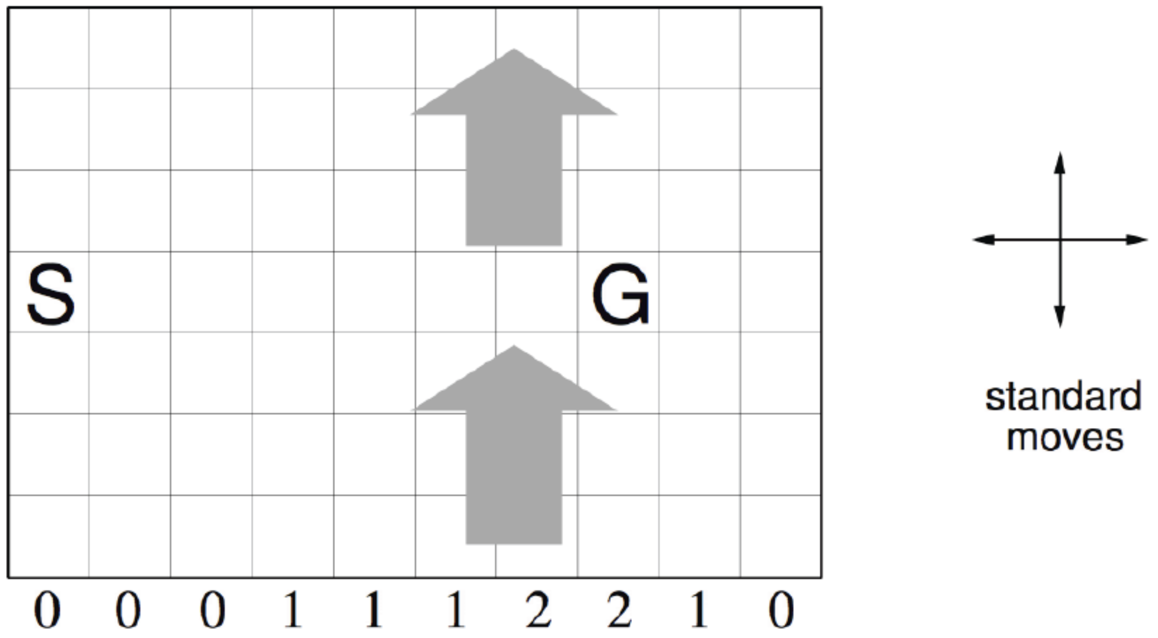


**Sarsa: An on-policy TD control algorithm**

```
Initialize $Q(s,a), \forall s \in \mathcal{S}, a \in \mathcal{A}(s)$, arbitrarily, and $Q(\textit{terminal-state}, \cdot) = 0$
Repeat (for each episode):
    Initialize $S$
    Choose $A$ from $S$ using policy derived from $Q$ (e.g., $\epsilon$-greedy)
    Repeat (for each step of episode):
        Take action $A$, observe $R$, $S'$
        Choose $A'$ from $S'$ using policy derived from $Q$ (e.g., $\epsilon$-greedy)
        $Q(S,A) \leftarrow Q(S,A) + \alpha\big[R + \gamma Q(S',A') - Q(S,A)\big]$
        $S \leftarrow S'$; $A \leftarrow A'$;
    until $S$ is terminal
```

- NOTE: on-policy TD control sample actions by the current policy, i.e., the two ‘A’s in SARSA are both chosen by the current policy

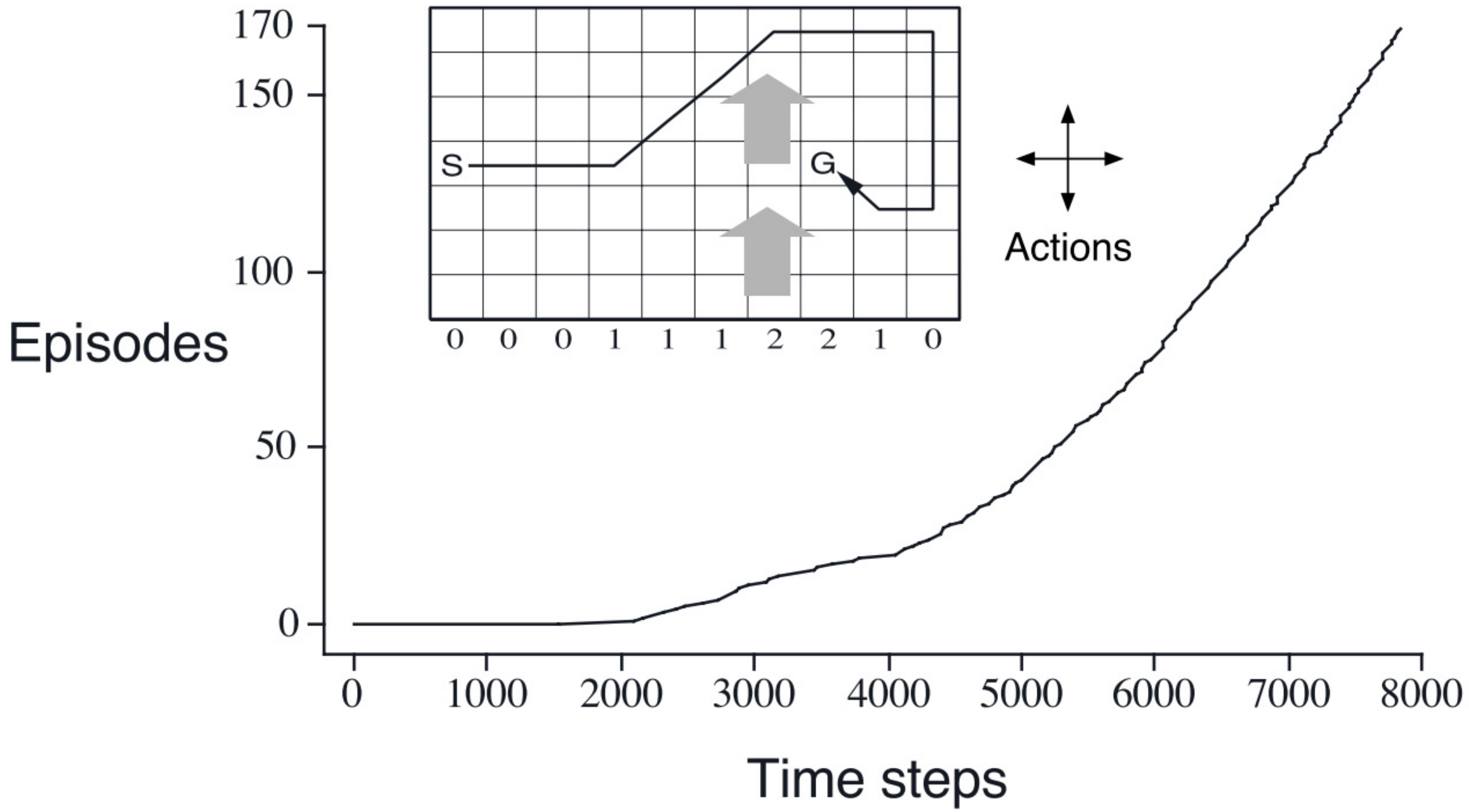
# SARSA Example: Windy Gridworld



- Reward = -1 per time-step until reaching goal
- Undiscounted

# SARSA Example: Windy Gridworld

optimal a trajectory



Note: as the training proceeds, the Sarsa policy achieves the goal more and more quickly

# Off-Policy Learning

- Evaluate target policy $\pi(a|s)$ to compute $V^\pi(s)$ or $Q^\pi(s,a)$
- While following behavior policy $\mu(a|s)$

$$\{s_1, a_1, r_2, s_2, a_2, \ldots, s_T\} \sim \mu$$

- Why off-policy learning is important?
  - Learn from observing humans or other agents
  - Re-use experience generated from old policies
  - Learn about optimal policy while following exploratory policy
  - Learn about multiple policies while following one policy
  - An example of my research in MSR Cambridge
    - Collective Noise Contrastive Estimation for Policy Transfer Learning. AAAI 2016.

# Q-Learning

- For off-policy learning of action-value *Q*(*s*,*a*)
- No importance sampling is required (why?)
- The next action is chosen using behavior policy $a_{t+1} \sim \mu(\cdot|s_t)$
- But we consider alternative successor action $a \sim \pi(\cdot|s_t)$
- And update $Q(s_t, a_t)$ towards value of alternative action

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_{t+1} + \gamma Q(s_{t+1}, \underset{\substack{\uparrow \\ \text{action} \\ \text{from } \pi \\ \text{not } \mu}}{a'}) - Q(s_t, a_t))$$

# Off-Policy Control with Q-Learning

- Allow both behavior and target policies to improve
- The target policy *π* is greedy w.r.t. *Q*(*s*,*a*)

$$\pi(s_{t+1}) = \arg\max_{a'} Q(s_{t+1}, a')$$

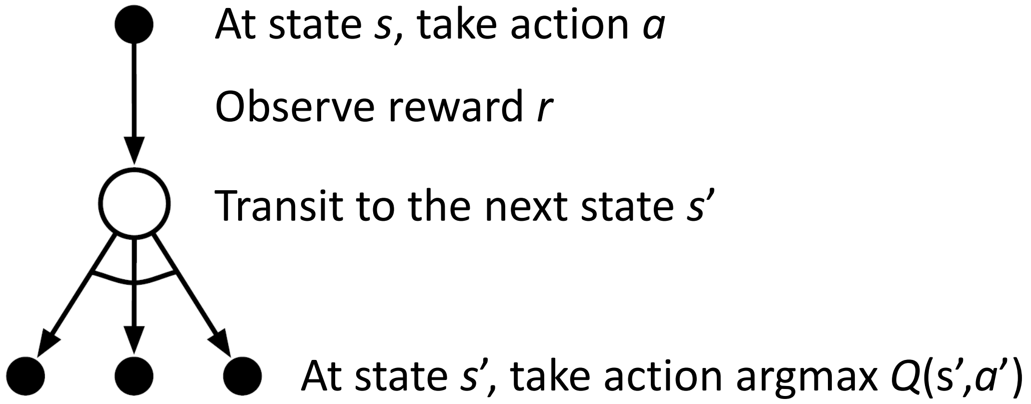
- The behavior policy *μ* is e.g. *ε*-greedy policy w.r.t. *Q*(*s*,*a*)
- The Q-learning target then simplifies

$$\begin{aligned} r_{t+1} + \gamma Q(s_{t+1}, a') &= r_{t+1} + \gamma Q(s_{t+1}, \arg\max_{a'} Q(s_{t+1}, a')) \\ &= r_{t+1} + \gamma \max_{a'} Q(s_{t+1}, a') \end{aligned}$$

- Q-learning update

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_{t+1} + \gamma \max_{a'} Q(s_{t+1}, a') - Q(s_t, a_t))$$

# Q-Learning Control Algorithm

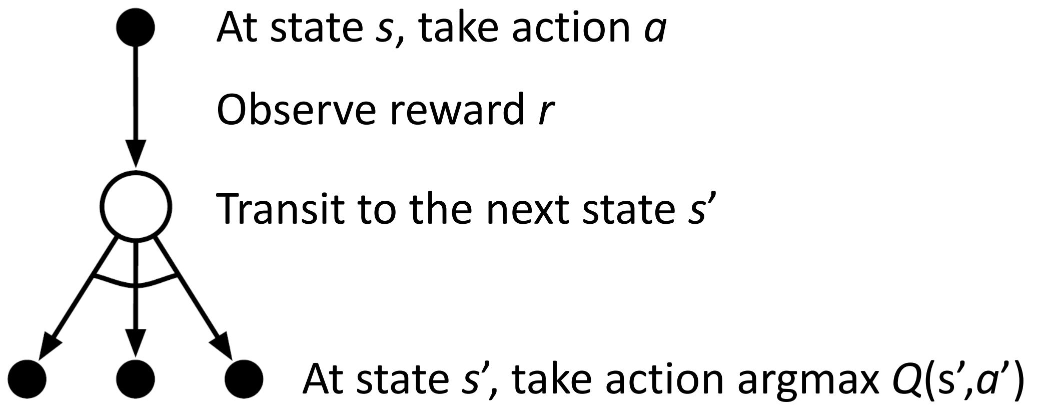

At state *s*, take action *a*
Observe reward *r*
Transit to the next state *s'*
At state *s'*, take action argmax *Q*(s',*a'*)

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_{t+1} + \gamma \max_{a'} Q(s_{t+1}, a') - Q(s_t, a_t))$$

- Theorem: Q-learning control converges to the optimal action-value function

$$Q(s, a) \rightarrow Q^*(s, a)$$

# Q-Learning Control Algorithm


At state *s*, take action *a*
Observe reward *r*
Transit to the next state *s'*
At state *s'*, take action argmax *Q*(s',*a'*)

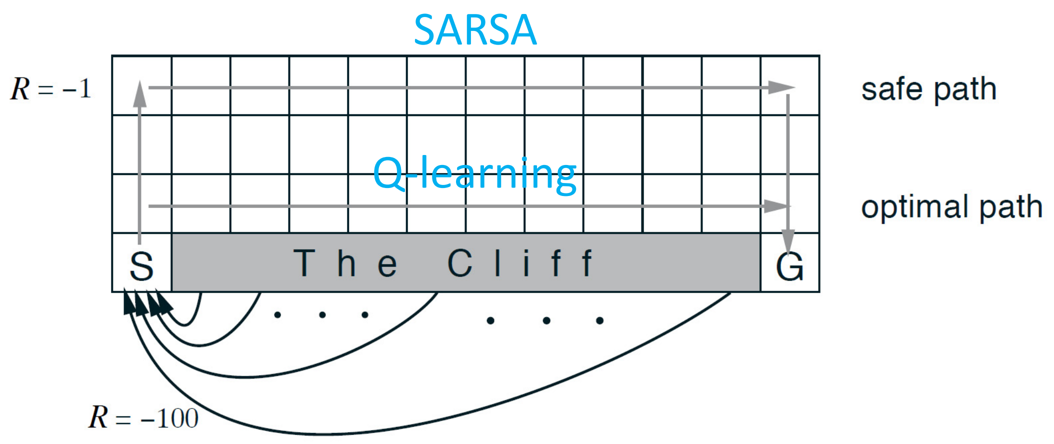
$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_{t+1} + \gamma \max_{a'} Q(s_{t+1}, a') - Q(s_t, a_t))$$

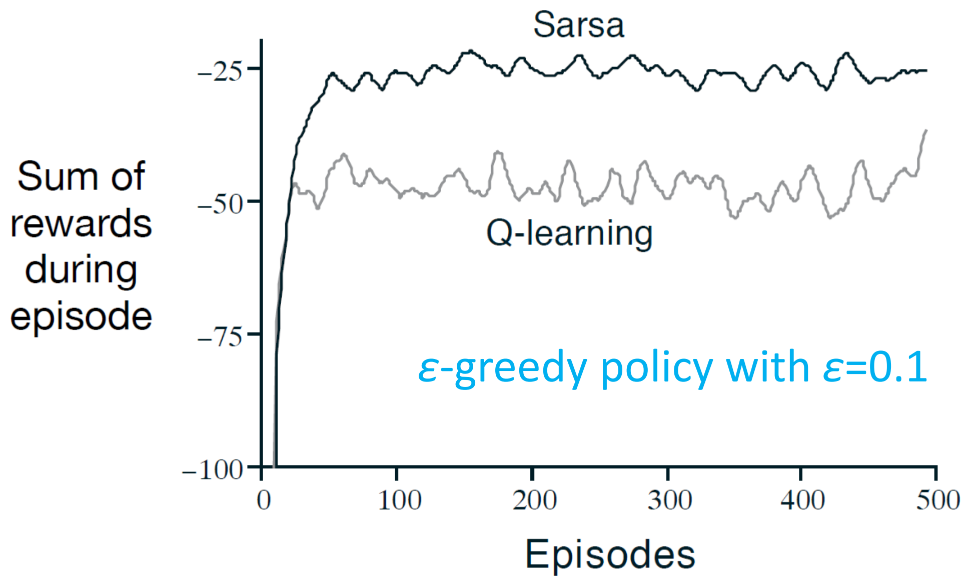
- Why Q-learning is an off-policy control method?
  - Learning from SARS generated by another policy *μ*
  - The first action *a* and the corresponding reward *r* are from *μ*
  - The next action *a'* is picked by the target policy

$$\pi(s_{t+1}) = \arg\max_{a'} Q(s_{t+1}, a')$$

# SARSA vs. Q-Learning Experiments

- Cliff-walking
    - Undiscounted reward
    - Episodic task
    - Reward = -1 on all transitions
    - Stepping into cliff area incurs -100 reward and sent the agent back to the start
- Why the results are like this?





PART II

# Approximation Methods in Reinforcement Learning

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Oct 14 2018, SJTU

# Review of What We have Learned

- Model-based dynamic programming
  - Value iteration $V(s) = R(s) + \max_{a \in A} \gamma \sum_{s' \in S} P_{sa}(s')V(s')$
  - Policy iteration $\pi(s) = \arg\max_{a \in A} \sum_{s' \in S} P_{sa}(s')V(s')$
- Model-free reinforcement learning
  - On-policy MC $V(s_t) \leftarrow V(s_t) + \alpha(G_t - V(s_t))$
  - On-policy TD $V(s_t) \leftarrow V(s_t) + \alpha(r_{t+1} + \gamma V(s_{t+1}) - V(s_t))$
  - On-policy TD SARSA

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_{t+1} + \gamma Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t))$$

  - Off-policy TD Q-learning

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_{t+1} + \gamma \max_{a'} Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t))$$

## Key Problem to Solve in This Lecture

- In all previous models, we have created a lookup table to maintain a variable $V(s)$ for each state or $Q(s,a)$ for each state-action

- What if we have a large MDP, i.e.
  - the state or state-action space is too large
  - or the state or action space is continuous

  to maintain $V(s)$ for each state or $Q(s,a)$ for each state-action?
  - For example
    - Game of Go ($10^{170}$ states)
    - Helicopter, autonomous car (continuous state space)

## Content

- Solutions for large MDPs
  - Discretize or bucketize states/actions
  - Build parametric value function approximation

- Policy gradient

- Deep reinforcement learning and multi-agent RL
  - Leave as future lectures

# Content

- Solutions for large MDPs
    - Discretize or bucketize states/actions
    - Build parametric value function approximation

- Policy gradient

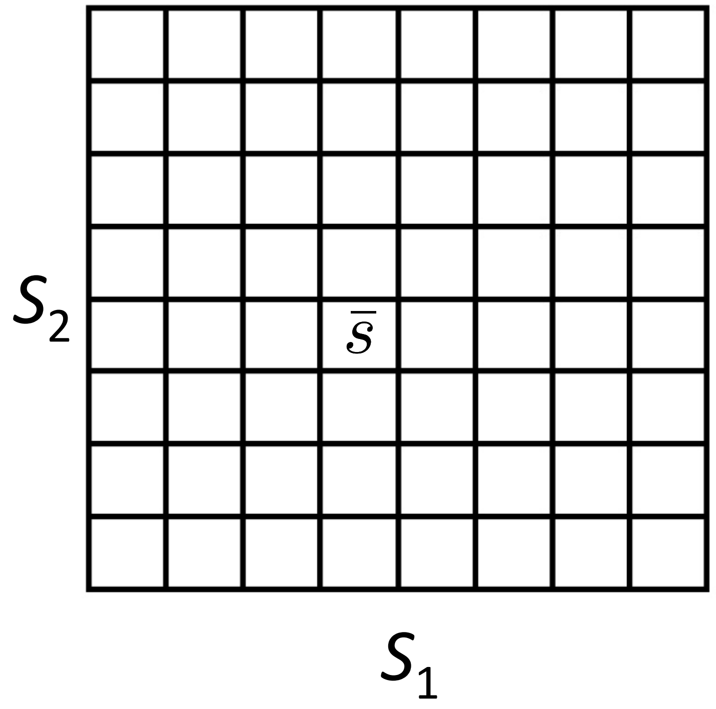
- Deep reinforcement learning and multi-agent RL
    - Leave as future lectures

# Discretization Continuous MDP

- For a continuous-state MDP, we can discretize the state space
  - For example, if we have 2D states ($s_1$, $s_2$), we can use a grid to discretize the state space
  - The discrete state $\bar{s}$
  - The discretized MDP:

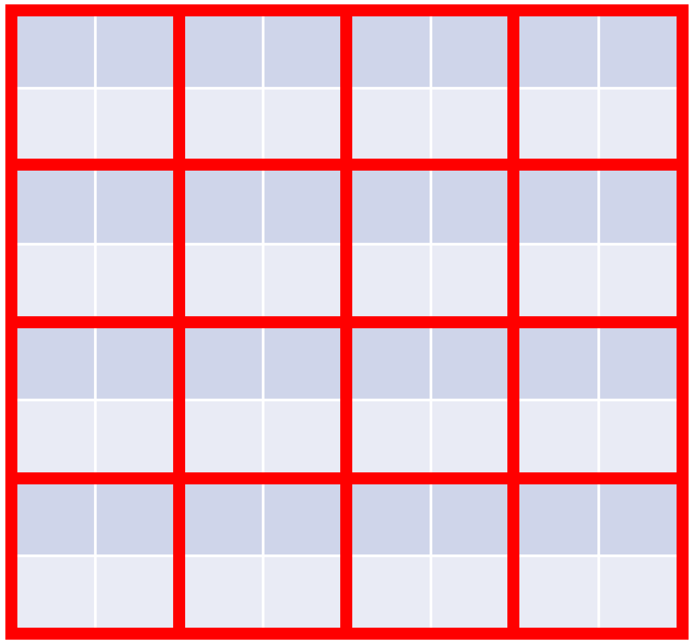
$$(\bar{S}, A, \{P_{\bar{s}a}\}, \gamma, R)$$

  - Then solve this MDP with any previous solutions



## Bucketize Large Discrete MDP

- For a large discrete-state MDP, we can bucketize the states to down sample the states
  - To use domain knowledge to merge similar discrete states
    - For example, clustering using state features extracted from domain knowledge

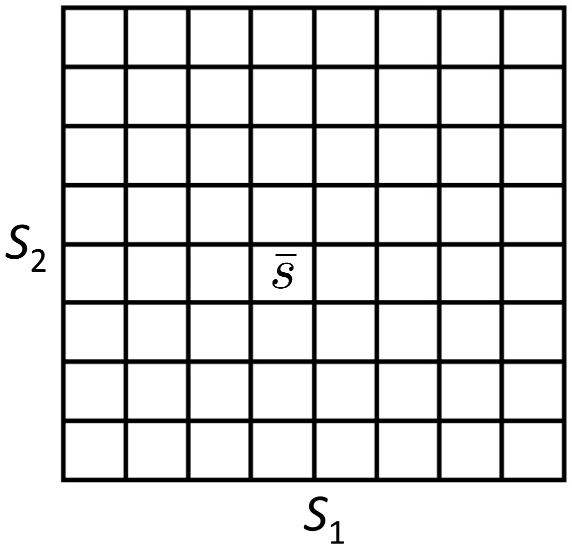


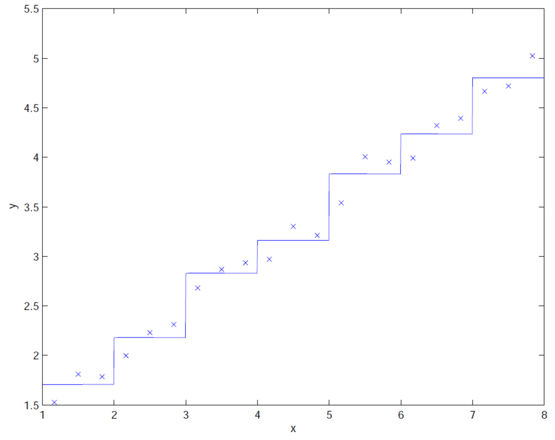
# Discretization/Bucketization

- Pros
  - Straightforward and off-the-shelf
  - Efficient
  - Can work well for many problems

- Cons
  - A fairly naïve representation for *V*
  - Assumes a constant value over each discretized cell
  - Curse of dimensionality

$$S = \mathbb{R}^n \Rightarrow \bar{S} = \{1, \ldots, k\}^n$$





## Parametric Value Function Approximation

- Create parametric (thus learnable) functions to approximate the value function

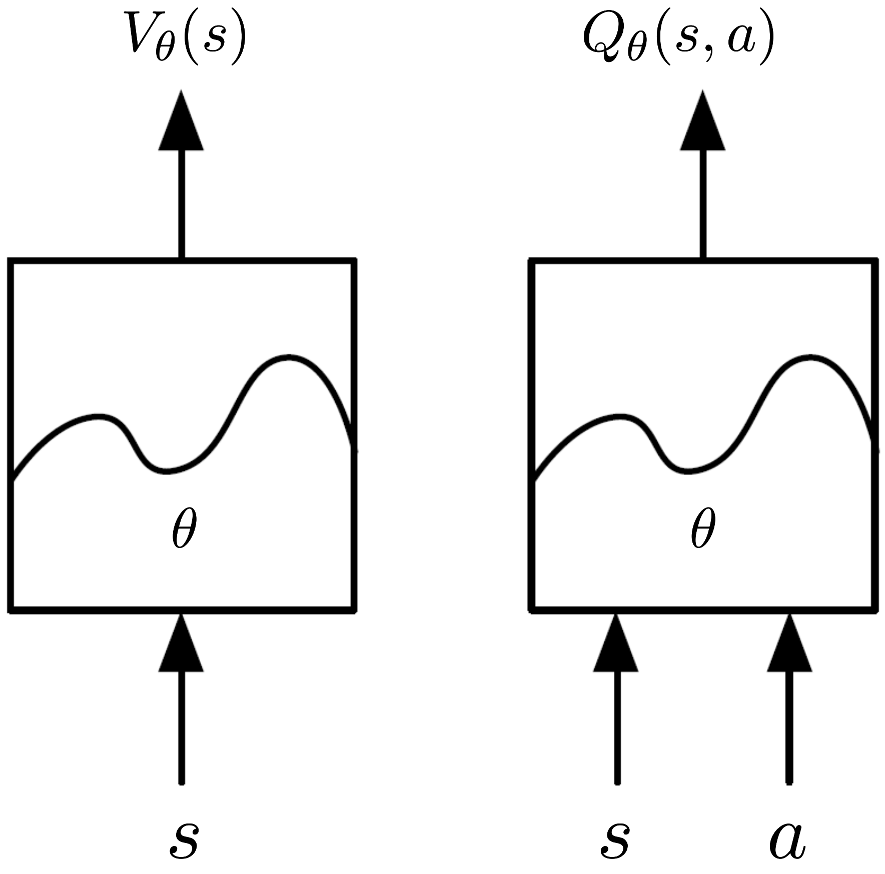
$$V_\theta(s) \simeq V^\pi(s)$$

$$Q_\theta(s,a) \simeq Q^\pi(s,a)$$

  - $\vartheta$ is the parameters of the approximation function, which can be updated by reinforcement learning

  - Generalize from seen states to unseen states

# Main Types of Value Function Approx.



Many function approximations

- (Generalized) linear model
- Neural network
- Decision tree
- Nearest neighbor
- Fourier / wavelet bases

Differentiable functions

- (Generalized) linear model
- Neural network

We assume the model is suitable to be trained for non-stationary, non-iid data

# Value Function Approx. by SGD

- Goal: find parameter vector $\vartheta$ minimizing mean-squared error between approximate value function $V_\vartheta(s)$ and true value $V^\pi(s)$

$$J(\theta) = \mathbb{E}_\pi\Big[\frac{1}{2}(V^\pi(s) - V_\theta(s))^2\Big]$$

- Gradient to minimize the error

$$-\frac{\partial J(\theta)}{\partial \theta} = \mathbb{E}_\pi\Big[(V^\pi(s) - V_\theta(s))\frac{\partial V_\theta(s)}{\partial \theta}\Big]$$

- Stochastic gradient descent on one sample

$$\begin{aligned}\theta &\leftarrow \theta - \alpha\frac{\partial J(\theta)}{\partial \theta}\\ &= \theta + \alpha(V^\pi(s) - V_\theta(s))\frac{\partial V_\theta(s)}{\partial \theta}\end{aligned}$$

# Featurize the State

- Represent state by a feature vector

$$x(s) = \begin{bmatrix} x_1(s) \\ \vdots \\ x_k(s) \end{bmatrix}$$

- For example of a helicopter
  - 3D location
  - 3D speed (differentiation of location)
  - 3D acceleration (differentiation of speed)

# Linear Value Function Approximation

- Represent value function by a linear combination of features

$$V_\theta(s) = \theta^\top x(s)$$

- Objective function is quadratic in parameters $\vartheta$

$$J(\theta) = \mathbb{E}_\pi\left[\frac{1}{2}(V^\pi(s) - \theta^\top x(s))^2\right]$$

- Thus stochastic gradient descent converges on global optimum

$$\begin{aligned}\theta &\leftarrow \theta - \alpha\frac{\partial J(\theta)}{\partial \theta}\\ &= \theta + \underset{\substack{\uparrow\\ \text{Step}\\ \text{size}}}{\alpha}(\underbrace{V^\pi(s) - V_\theta(s)}_{\substack{\uparrow\\ \text{Prediction}\\ \text{error}}})\underset{\substack{\uparrow\\ \text{Feature}\\ \text{value}}}{x(s)}\end{aligned}$$

# Monte-Carlo with Value Function Approx.

$$\theta \leftarrow \theta + \alpha(V^{\pi}(s) - V_{\theta}(s))x(s)$$

- Now we specify the target value function $V^{\pi}(s)$
- We can apply supervised learning to "training data"

$$\langle s_1, G_1 \rangle, \langle s_2, G_2 \rangle, \ldots, \langle s_T, G_T \rangle$$

- For each data instance $<s_t, G_t>$

$$\theta \leftarrow \theta + \alpha(G_t - V_{\theta}(s))x(s_t)$$

- MC evaluation at least converges to a local optimum
  - In linear case it converges to a global optimum

# TD Learning with Value Function Approx.

$$\theta \leftarrow \theta + \alpha(V^\pi(s) - V_\theta(s))x(s)$$

- TD target $r_{t+1} + \gamma V_\theta(s_{t+1})$ is a biased sample of true target value $V^\pi(s_t)$
- Supervised learning from “training data”

$$\langle s_1, r_2 + \gamma V_\theta(s_2)\rangle, \langle s_2, r_3 + \gamma V_\theta(s_3)\rangle, \ldots, \langle s_T, r_T\rangle$$

- For each data instance $\langle s_t, r_{t+1} + \gamma V_\theta(s_{t+1})\rangle$

$$\theta \leftarrow \theta + \alpha(r_{t+1} + \gamma V_\theta(s_{t+1}) - V_\theta(s))x(s_t)$$

- Linear TD converges (close) to global optimum

# Action-Value Function Approximation

- Approximate the action-value function

$$Q_\theta(s,a) \simeq Q^\pi(s,a)$$

- Minimize mean squared error

$$J(\theta) = \mathbb{E}_\pi\Big[\frac{1}{2}(Q^\pi(s,a) - Q_\theta(s,a))^2\Big]$$

- Stochastic gradient descent on one sample

$$\begin{aligned}\theta &\leftarrow \theta - \alpha\frac{\partial J(\theta)}{\partial\theta} \\ &= \theta + \alpha(Q^\pi(s,a) - Q_\theta(s,a))\frac{\partial Q_\theta(s,a)}{\partial\theta}\end{aligned}$$

# Linear Action-Value Function Approx.

- Represent state-action pair by a feature vector

$$x(s,a) = \begin{bmatrix} x_1(s,a) \\ \vdots \\ x_k(s,a) \end{bmatrix}$$

- Parametric Q function, e.g., the linear case

$$Q_\theta(s,a) = \theta^\top x(s,a)$$

- Stochastic gradient descent update

$$\begin{aligned} \theta &\leftarrow \theta - \alpha \frac{\partial J(\theta)}{\partial \theta} \\ &= \theta + \alpha (Q^\pi(s,a) - \theta^\top x(s,a)) x(s,a) \end{aligned}$$

# TD Learning with Value Function Approx.

$$\theta \leftarrow \theta + \alpha(Q^{\pi}(s,a) - Q_\theta(s,a))\frac{\partial Q_\theta(s,a)}{\partial \theta}$$

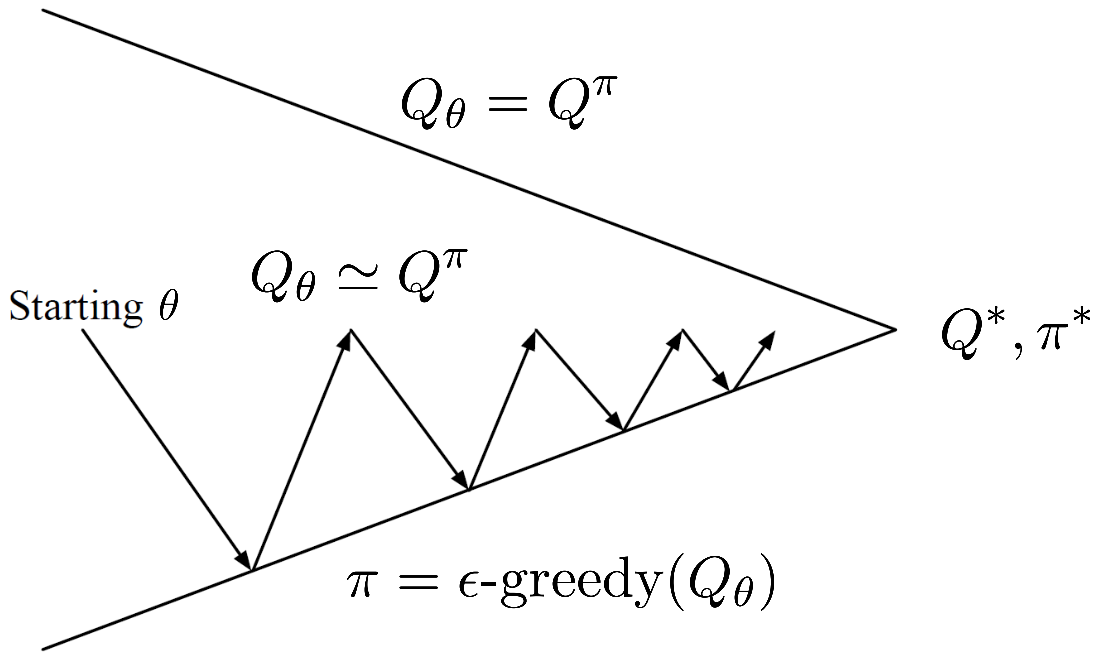
- For MC, the target is the return $G_t$

$$\theta \leftarrow \theta + \alpha(G_t - Q_\theta(s,a))\frac{\partial Q_\theta(s,a)}{\partial \theta}$$

- For TD, the target is $r_{t+1} + \gamma Q_\theta(s_{t+1}, a_{t+1})$

$$\theta \leftarrow \theta + \alpha(r_{t+1} + \gamma Q_\theta(s_{t+1}, a_{t+1}) - Q_\theta(s,a))\frac{\partial Q_\theta(s,a)}{\partial \theta}$$

# Control with Value Function Approx.



- Policy evaluation: approximately policy evaluation $Q_\theta \simeq Q^\pi$
- Policy improvement: *ε*-greedy policy improvement

# NOTE of TD Update

- The TD target is
  - State value

$$\theta \leftarrow \theta + \alpha(V^{\pi}(s_t) - V_\theta(s_t))\frac{\partial V_\theta(s_t)}{\partial \theta}$$

$$= \theta + \alpha(r_{t+1} + \gamma V_\theta(s_{t+1}) - V_\theta(s))\frac{\partial V_\theta(s_t)}{\partial \theta}$$

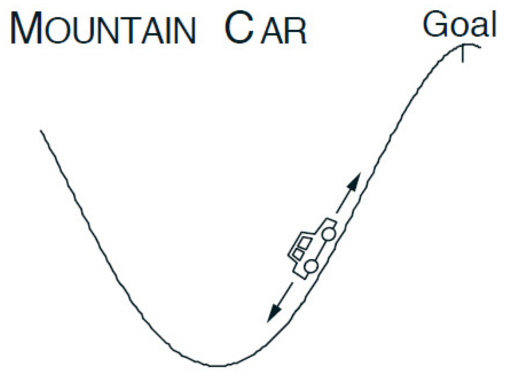
  - Action value

$$\theta \leftarrow \theta + \alpha(Q^{\pi}(s,a) - Q_\theta(s,a))\frac{\partial Q_\theta(s,a)}{\partial \theta}$$

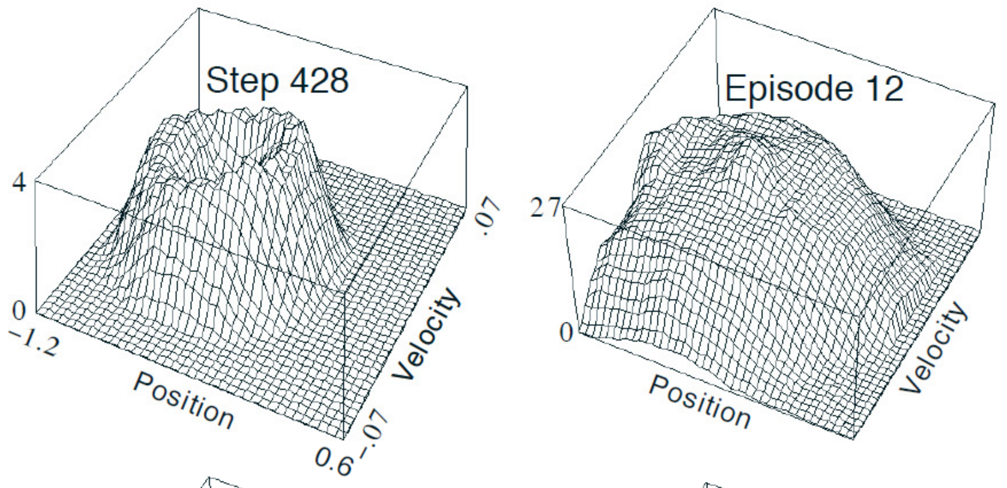
$$= \theta + \alpha(r_{t+1} + \gamma Q_\theta(s_{t+1}, a_{t+1}) - Q_\theta(s,a))\frac{\partial Q_\theta(s,a)}{\partial \theta}$$

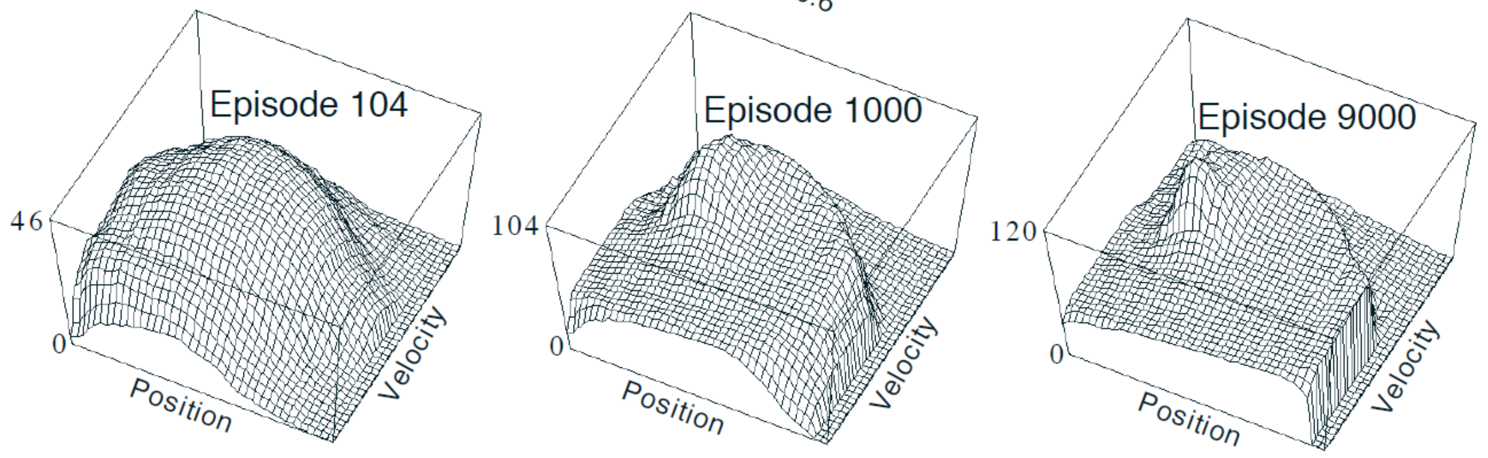
- Although $\vartheta$ is in the TD target, we don't calculate gradient from the target. Think about why.

# Case Study: Mountain Car



The gravity is stronger than the car's engine

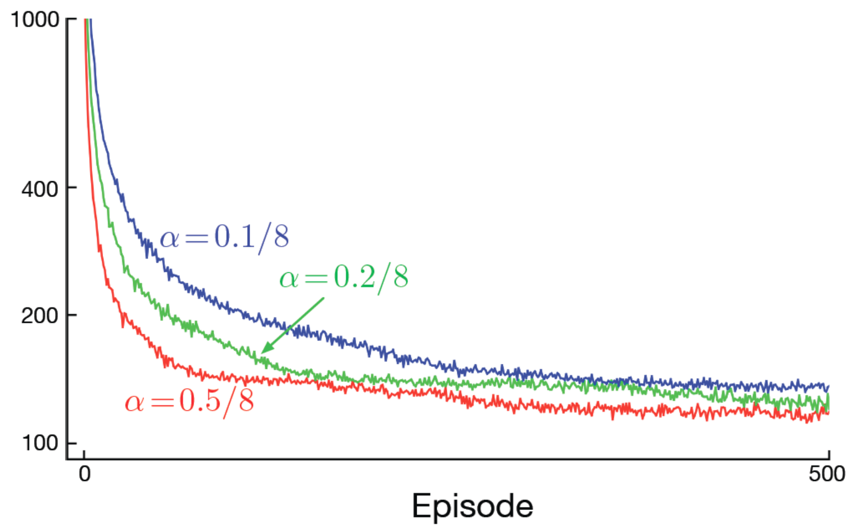




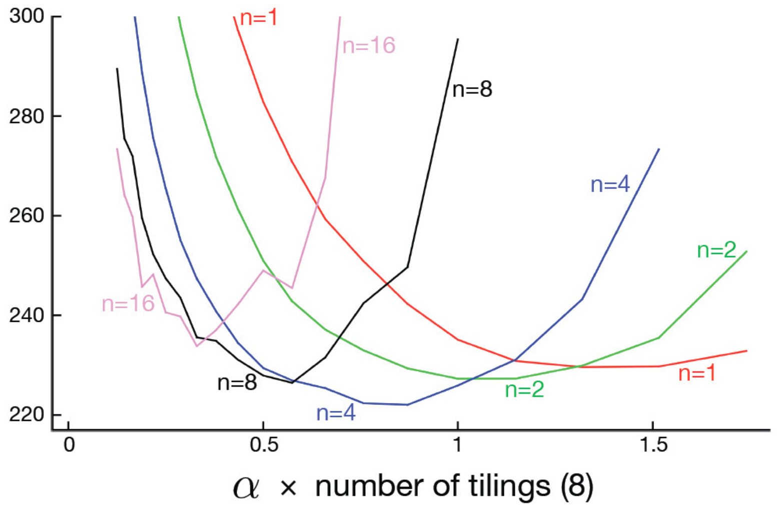
Cost-to-go function

# Case Study: Mountain Car

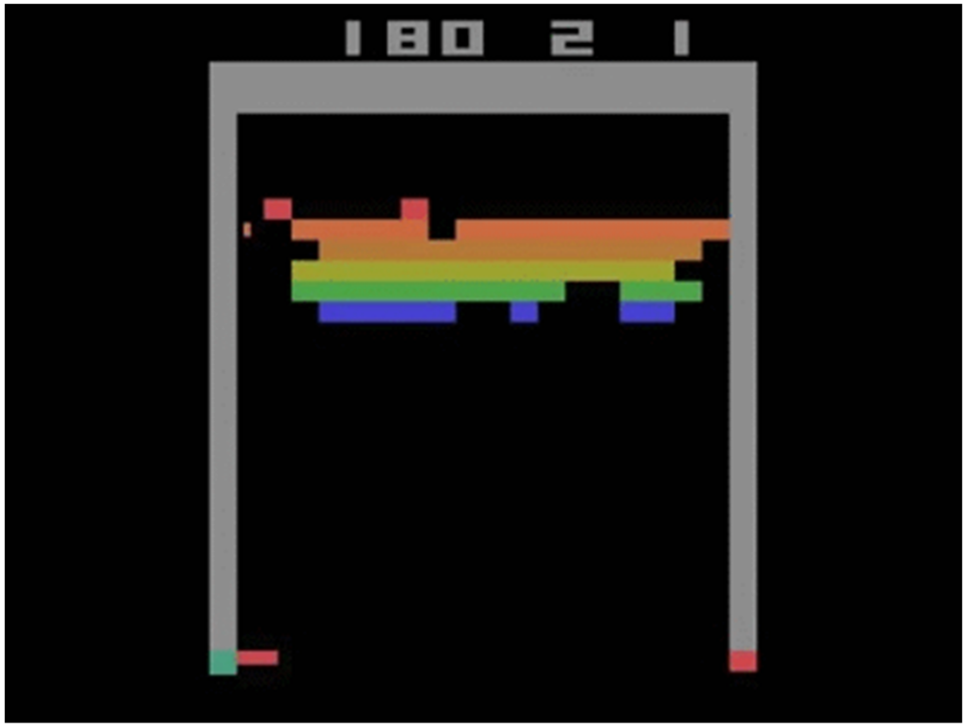
Mountain Car
Steps per episode
log scale
averaged over 100 runs

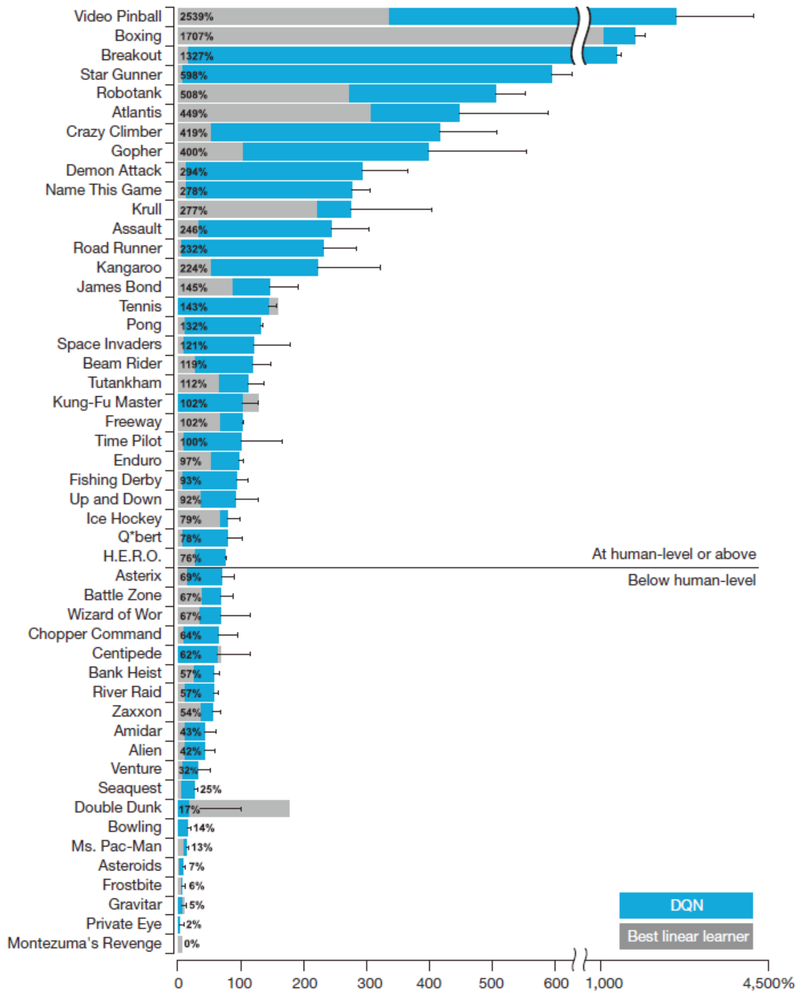


Mountain Car
Steps per episode
averaged over
first 50 episodes
and 100 runs



# Deep Q-Network (DQN)

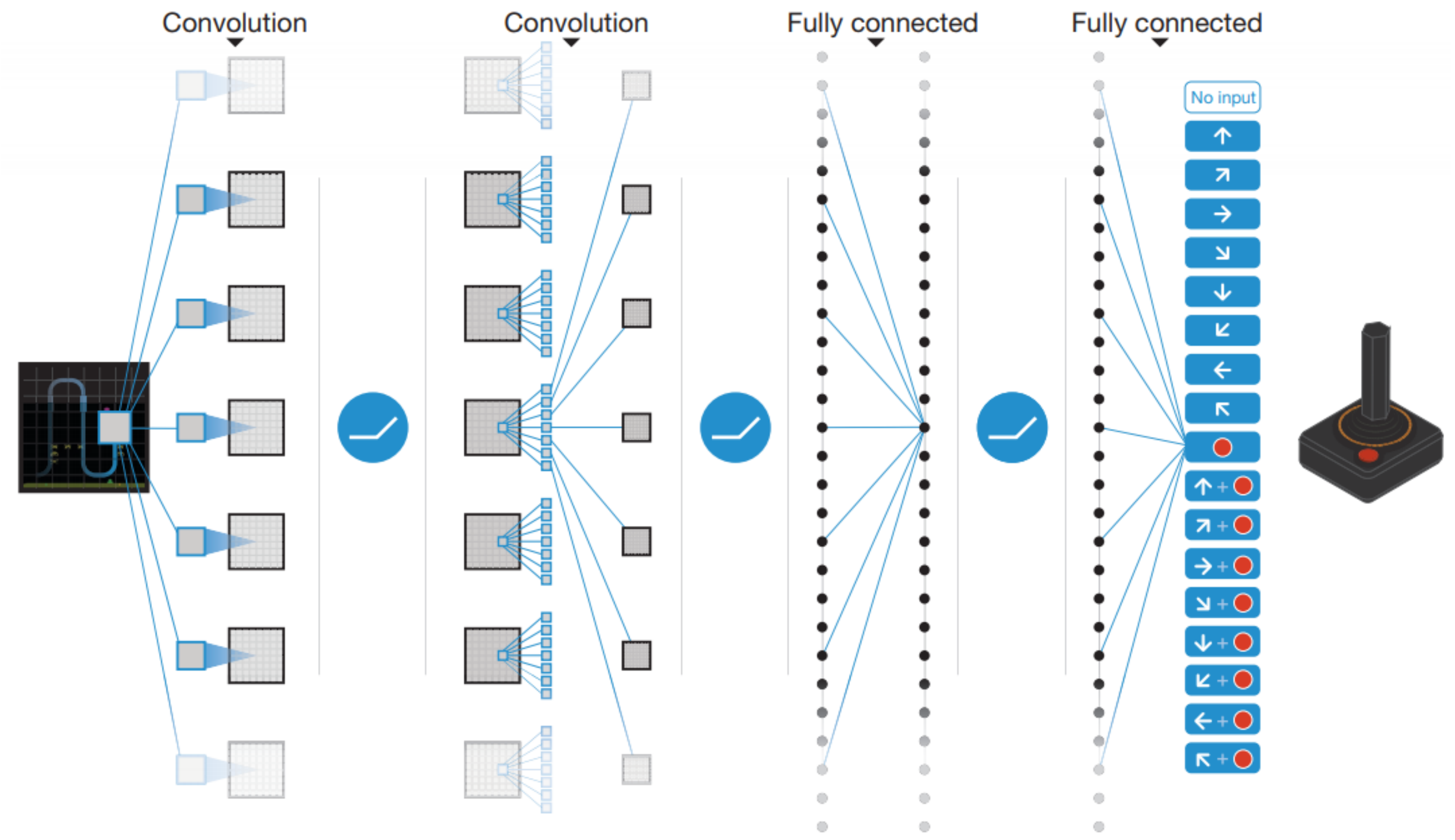




Volodymyr Mnih, Koray Kavukcuoglu, David Silver et al. Playing Atari with Deep Reinforcement Learning. NIPS 2013 workshop.
Volodymyr Mnih, Koray Kavukcuoglu, David Silver et al. Human-level control through deep reinforcement learning. Nature 2015.

# Deep Q-Network (DQN)

- Implement Q function with deep neural network



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Volodymyr Mnih, Koray Kavukcuoglu, David Silver et al. Human-level control through deep reinforcement learning. Nature 2015.

# Deep Q-Network (DQN)

- The loss function of Q-learning update at iteration *i*

$$L_i(\theta_i) = \mathbb{E}_{(s,a,r,s')\sim U(D)}\Big[\big(\underbrace{r + \gamma \max_{a'} Q(s',a';\theta_i^-)}_{\text{target Q value}} - \underbrace{Q(s,a;\theta_i)}_{\text{estimated Q value}}\big)^2\Big]$$

  - $\vartheta_i$ are the network parameters to be updated at iteration *i*
    - Updated with standard back-propagation algorithms
  - $\vartheta_i^-$ are the target network parameters
    - Only updated with $\vartheta_i$ for every *C* steps
  - (*s*,*a*,*r*,*s'*)~*U*(*D*): the samples are uniformly drawn from the experience pool *D*
    - Thus to avoid the overfitting to the recent experiences

Volodymyr Mnih, Koray Kavukcuoglu, David Silver et al. Playing Atari with Deep Reinforcement Learning. NIPS 2013 workshop.

Volodymyr Mnih, Koray Kavukcuoglu, David Silver et al. Human-level control through deep reinforcement learning. Nature 2015.

# Content

- Solutions for large MDPs
  - Discretize or bucketize states/actions
  - Build parametric value function approximation
- Policy gradient
- Deep reinforcement learning and multi-agent RL

## Parametric Policy

- We can parametrize the policy

$$\pi_\theta(a|s)$$

which could be deterministic

$$a = \pi_\theta(s)$$

or stochastic

$$\pi_\theta(a|s) = P(a|s;\theta)$$

  - $\vartheta$ is the parameters of the policy
  - Generalize from seen states to unseen states
  - We focus on model-free reinforcement learning

## Policy-based RL

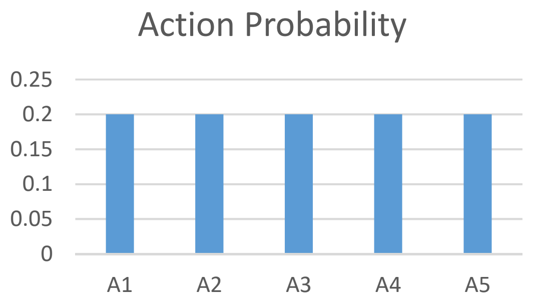
- Advantages
  - Better convergence properties
  - Effective in high-dimensional or continuous action spaces
    - No.1 reason: for value function, you have to take a max operation
  - Can learn stochastic polices

- Disadvantages
  - Typically converge to a local rather than global optimum
  - Evaluating a policy is typically inefficient and of high variance

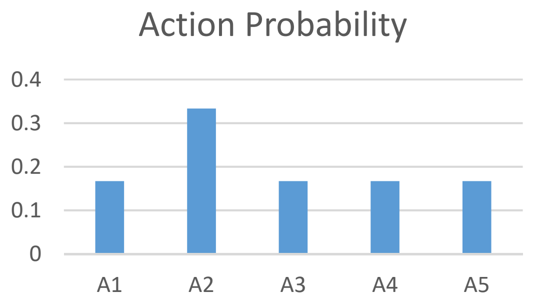
# Policy Gradient

- For stochastic policy $\pi_\theta(a|s) = P(a|s;\theta)$
- Intuition
  - lower the probability of the action that leads to low value/reward
  - higher the probability of the action that leads to high value/reward
- A 5-action example

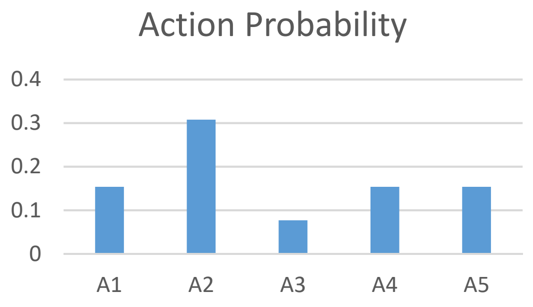
1. Initialize $\vartheta$



3. Update $\vartheta$ by policy gradient



5. Update $\vartheta$ by policy gradient



2. Take action A2
Observe positive reward

4. Take action A3
Observe negative reward

## Policy Gradient in One-Step MDPs

- Consider a simple class of one-step MDPs
  - Starting in state $s \sim d(s)$
  - Terminating after one time-step with reward $r_{sa}$

- Policy expected value

$$J(\theta) = \mathbb{E}_{\pi_\theta}[r] = \sum_{s \in S} d(s) \sum_{a \in A} \pi_\theta(a|s) r_{sa}$$

$$\frac{\partial J(\theta)}{\partial \theta} = \sum_{s \in S} d(s) \sum_{a \in A} \frac{\partial \pi_\theta(a|s)}{\partial \theta} r_{sa}$$

# Likelihood Ratio

- Likelihood ratios exploit the following identity

$$\frac{\partial \pi_\theta(a|s)}{\partial \theta} = \pi_\theta(a|s) \frac{1}{\pi_\theta(a|s)} \frac{\partial \pi_\theta(a|s)}{\partial \theta}$$

$$= \pi_\theta(a|s) \frac{\partial \log \pi_\theta(a|s)}{\partial \theta}$$

- Thus the policy's expected value

$$J(\theta) = \mathbb{E}_{\pi_\theta}[r] = \sum_{s \in S} d(s) \sum_{a \in A} \pi_\theta(a|s) \boxed{r_{sa}}$$

$$\frac{\partial J(\theta)}{\partial \theta} = \sum_{s \in S} d(s) \sum_{a \in A} \frac{\partial \pi_\theta(a|s)}{\partial \theta} r_{sa}$$

$$= \sum_{s \in S} d(s) \sum_{a \in A} \pi_\theta(a|s) \boxed{\frac{\partial \log \pi_\theta(a|s)}{\partial \theta} r_{sa}}$$

$$= \mathbb{E}_{\pi_\theta}\left[\frac{\partial \log \pi_\theta(a|s)}{\partial \theta} r_{sa}\right]$$

This can be approximated by sampling state *s* from *d*(*s*) and action *a* from $\pi_\vartheta$

# Policy Gradient Theorem

- The policy gradient theorem generalizes the likelihood ratio approach to multi-step MDPs
  - Replaces instantaneous reward $r_{sa}$ with long-term value $Q^{\pi_\theta}(s,a)$
- Policy gradient theorem applies to
  - start state objective $J_1$, average reward objective $J_{avR}$, and average value objective $J_{avV}$
- Theorem
  - For any differentiable policy $\pi_\theta(a|s)$, for any of policy objective function $J = J_1, J_{avR}, J_{avV}$, the policy gradient is

$$\frac{\partial J(\theta)}{\partial \theta} = \mathbb{E}_{\pi_\theta}\Big[\frac{\partial \log \pi_\theta(a|s)}{\partial \theta} Q^{\pi_\theta}(s,a)\Big]$$

Please refer to appendix of the slides for detailed proofs

## Monte-Carlo Policy Gradient (REINFORCE)

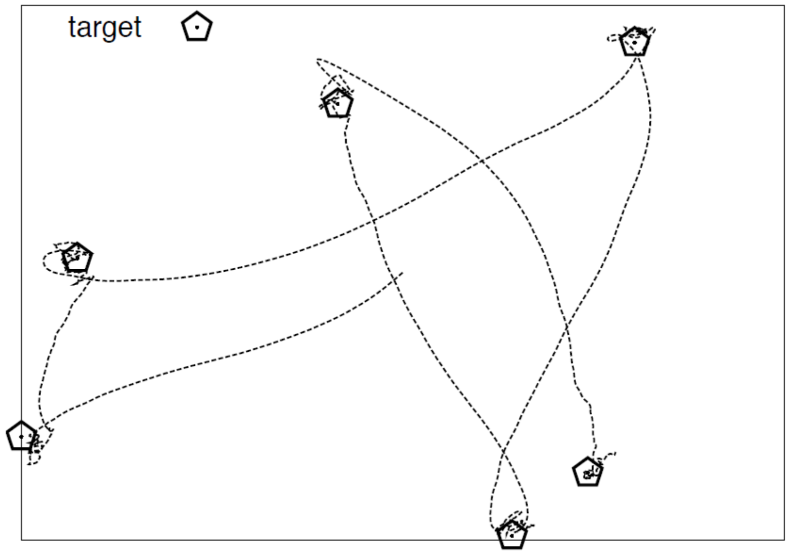
- Update parameters by stochastic gradient ascent
- Using policy gradient theorem
- Using return $G_t$ as an unbiased sample of $Q^{\pi_\theta}(s,a)$

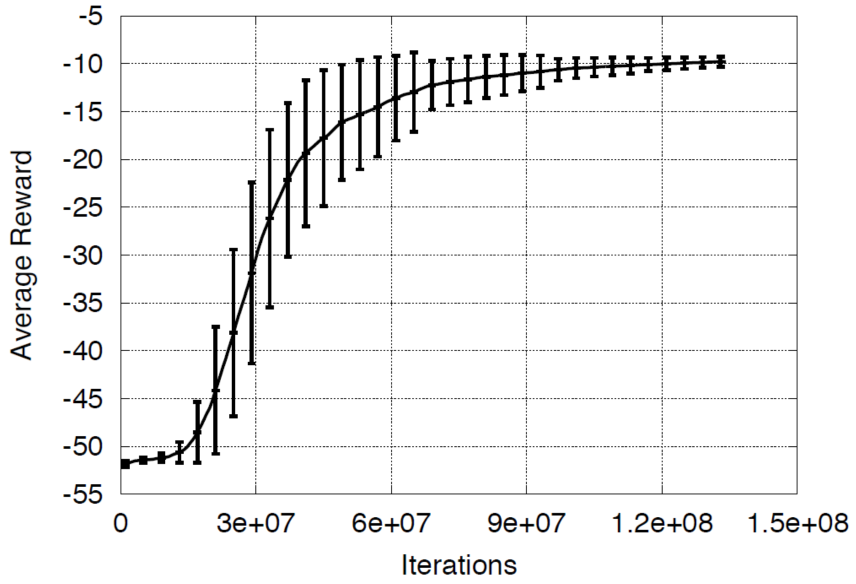
$$\Delta\theta_t = \alpha \frac{\partial \log \pi_\theta(a_t|s_t)}{\partial\theta} G_t$$

- REINFORCE Algorithm

```
Initialize ϑ arbitrarily
for each episode $\{s_1, a_1, r_2, \ldots, s_{T-1}, a_{T-1}, r_T\} \sim \pi_\theta$ do
    for t=1 to T-1 do
        $\theta \leftarrow \theta + \alpha \frac{\partial}{\partial\theta} \log \pi_\theta(a_t|s_t) G_t$
    end for
end for
return ϑ
```

# Puck World Example

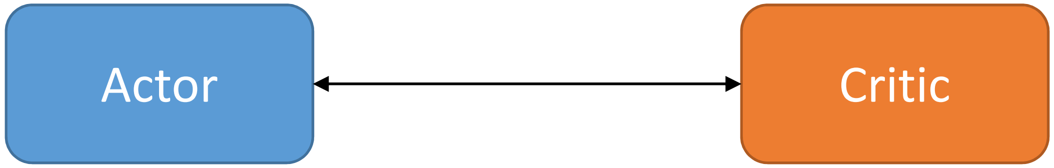




- Continuous actions exert small force on puck
- Puck is rewarded for getting close to target
- Target location is reset every 30 seconds
- Policy is trained using variant of MC policy gradient

# Actor-Critic Policy Gradient

- Actor-critic settings
  - Actor: a stochastic policy $\pi_\theta(a|s)$
  - Critic: a value estimator $Q_\phi(s,a)$



- Train the policy to maximize the estimated value from the critic

$$\max_\theta \mathbb{E}_{\pi_\theta}\Big[\pi_\theta(a|s)Q_\phi(s,a)\Big]$$

- Train action value function to minimize the estimation square error

$$\min_\phi \mathbb{E}_{\pi_\theta}\Big[\frac{1}{2}(r+\gamma Q_\phi(s',a')-Q_\phi(s,a))^2\Big]$$

## Summary

- Reinforcement learning
  - Decision making machine learning
  - Learning from trial-and-error interactions

- Model-based and model-free RL

- On-policy and off-policy model-free RL

- Value-based, policy-based and actor-critic RL