

2018 CS420, Machine Learning, Lecture 12

# Approximation Methods in Reinforcement Learning

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<http://wnzhang.net/teaching/cs420/index.html>

# Reinforcement Learning Materials

Our course on RL is mainly based on the materials from these masters.



Prof. Richard Sutton

- University of Alberta, Canada
- <http://incompleteideas.net/sutton/index.html>
- Reinforcement Learning: An Introduction (2<sup>nd</sup> edition)
- <http://incompleteideas.net/sutton/book/the-book-2nd.html>



Dr. David Silver

- Google DeepMind and UCL, UK
- <http://www0.cs.ucl.ac.uk/staff/d.silver/web/Home.html>
- UCL Reinforcement Learning Course
- <http://www0.cs.ucl.ac.uk/staff/d.silver/web/Teaching.html>



Prof. Andrew Ng

- Stanford University, US
- <http://www.andrewng.org/>
- Machine Learning (CS229) Lecture Notes 12: RL
- <http://cs229.stanford.edu/materials.html>

# Last Lecture

- Model-based dynamic programming

- Value iteration  $V(s) = R(s) + \max_{a \in A} \gamma \sum_{s' \in S} P_{sa}(s')V(s')$

- Policy iteration  $\pi(s) = \arg \max_{a \in A} \sum_{s' \in S} P_{sa}(s')V(s')$

- Model-free reinforcement learning

- On-policy MC  $V(s_t) \leftarrow V(s_t) + \alpha(G_t - V(s_t))$

- On-policy TD  $V(s_t) \leftarrow V(s_t) + \alpha(r_{t+1} + \gamma V(s_{t+1}) - V(s_t))$

- On-policy TD SARSA

- $Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_{t+1} + \gamma Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t))$

- Off-policy TD Q-learning

- $Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_{t+1} + \gamma \max_{a'} Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t))$

# Key Problem to Solve in This Lecture

- In all previous models, we have created a lookup table to maintain a variable  $V(s)$  for each state or  $Q(s,a)$  for each state-action
- What if we have a large MDP, i.e.
  - the state or state-action space is too large
  - or the state or action space is continuousto maintain  $V(s)$  for each state or  $Q(s,a)$  for each state-action?
- For example
  - Game of Go ( $10^{170}$  states)
  - Helicopter, autonomous car (continuous state space)

# Content

- Solutions for large MDPs
  - Discretize or bucketize states/actions
  - Build parametric value function approximation
- Policy gradient
- Deep reinforcement learning and multi-agent RL

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# Discretization Continuous MDP

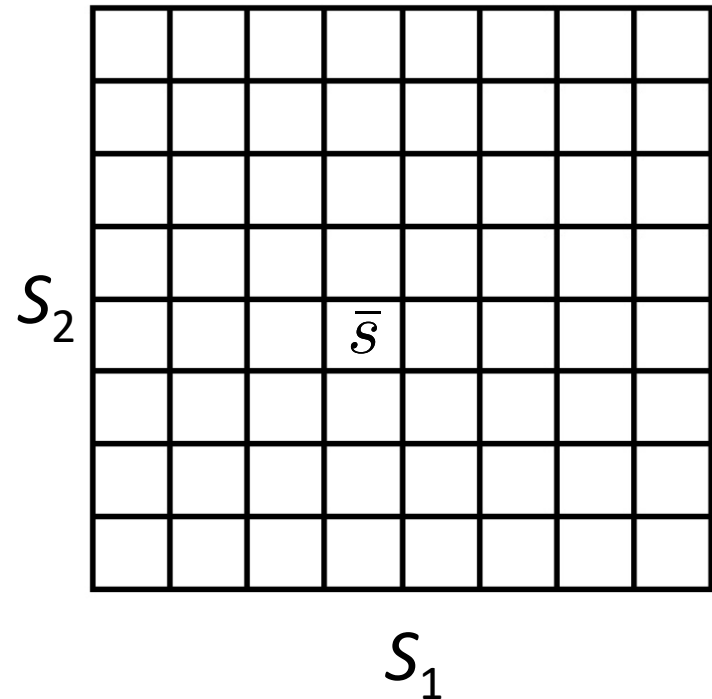
- For a continuous-state MDP, we can discretize the state space

- For example, if we have 2D states  $(s_1, s_2)$ , we can use a grid to discretize the state space

- The discrete state  $\bar{s}$
- The discretized MDP:

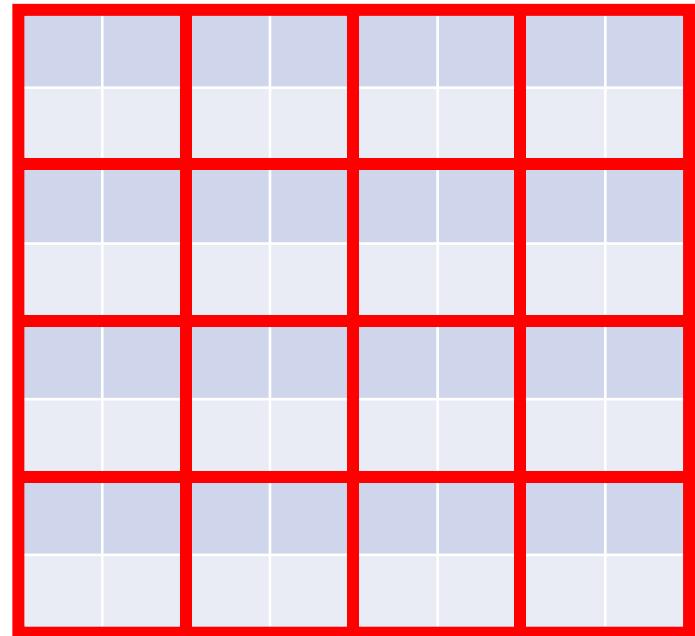
$$(\bar{S}, A, \{P_{\bar{s}a}\}, \gamma, R)$$

- Then solve this MDP with any previous solutions



# Bucketize Large Discrete MDP

- For a large discrete-state MDP, we can bucketize the states to down sample the states
  - To use domain knowledge to merge similar discrete states
    - For example, clustering using state features extracted from domain knowledge

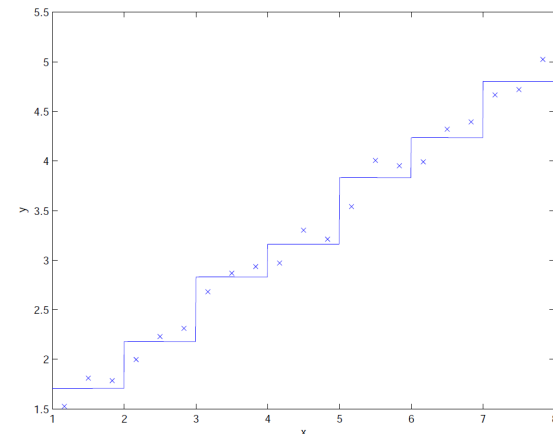
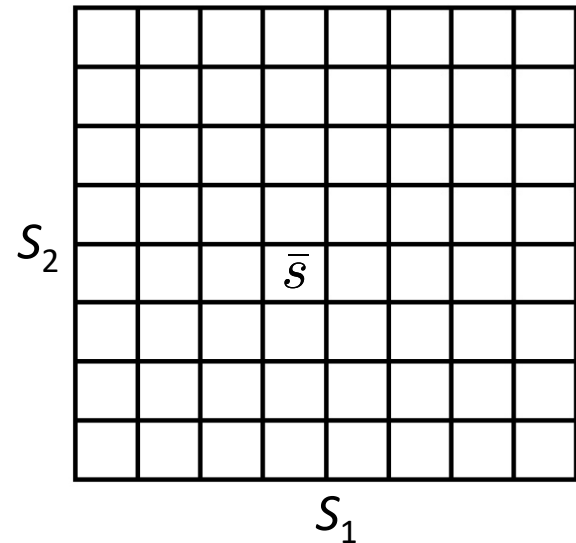




# Discretization/ Bucketization

- Pros
  - Straightforward and off-the-shelf
  - Efficient
  - Can work well for many problems
- Cons
  - A fairly naïve representation for  $V$
  - Assumes a constant value over each discretized cell
  - Curse of dimensionality

$$S = \mathbb{R}^n \Rightarrow \bar{S} = \{1, \dots, k\}^n$$



# Parametric Value Function Approximation

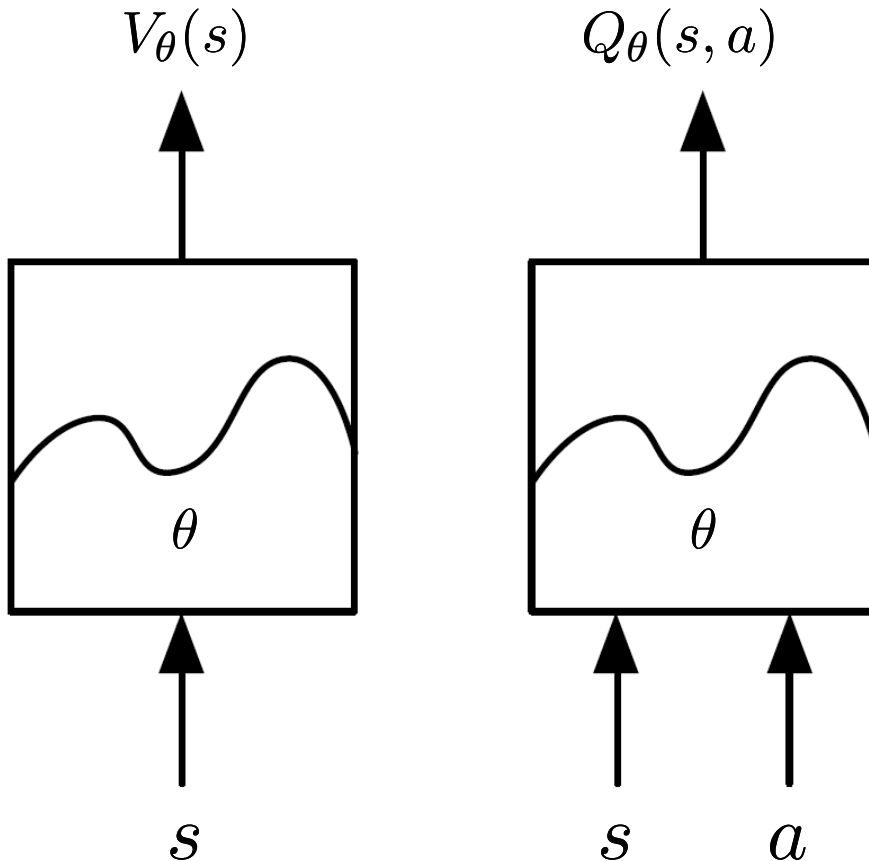
- Create parametric (thus learnable) functions to approximate the value function

$$V_{\theta}(s) \simeq V^{\pi}(s)$$

$$Q_{\theta}(s, a) \simeq Q^{\pi}(s, a)$$

- $\vartheta$  is the parameters of the approximation function, which can be updated by reinforcement learning
- Generalize from seen states to unseen states

# Main Types of Value Function Approx.



Many function approximations

- (Generalized) linear model
- Neural network
- Decision tree
- Nearest neighbor
- Fourier / wavelet bases

Differentiable functions

- (Generalized) linear model
- Neural network

We assume the model is suitable to be trained for non-stationary, non-iid data

# Value Function Approx. by SGD

- Goal: find parameter vector  $\vartheta$  minimizing mean-squared error between approximate value function  $V_{\vartheta}(s)$  and true value  $V^{\pi}(s)$

$$J(\theta) = \mathbb{E}_{\pi} \left[ \frac{1}{2} (V^{\pi}(s) - V_{\theta}(s))^2 \right]$$

- Gradient to minimize the error

$$-\frac{\partial J(\theta)}{\partial \theta} = \mathbb{E}_{\pi} \left[ (V^{\pi}(s) - V_{\theta}(s)) \frac{\partial V_{\theta}(s)}{\partial \theta} \right]$$

- Stochastic gradient descent on one sample

$$\begin{aligned} \theta &\leftarrow \theta - \alpha \frac{\partial J(\theta)}{\partial \theta} \\ &= \theta + \alpha (V^{\pi}(s) - V_{\theta}(s)) \frac{\partial V_{\theta}(s)}{\partial \theta} \end{aligned}$$

# Featurize the State

- Represent state by a feature vector

$$x(s) = \begin{bmatrix} x_1(s) \\ \vdots \\ x_k(s) \end{bmatrix}$$

- For example of a helicopter
  - 3D location
  - 3D speed (differentiation of location)
  - 3D acceleration (differentiation of speed)



# Linear Value Function Approximation

- Represent value function by a linear combination of features

$$V_{\theta}(s) = \theta^{\top} x(s)$$

- Objective function is quadratic in parameters  $\theta$

$$J(\theta) = \mathbb{E}_{\pi} \left[ \frac{1}{2} (V^{\pi}(s) - \theta^{\top} x(s))^2 \right]$$

- Thus stochastic gradient descent converges on global optimum

$$\begin{aligned} \theta &\leftarrow \theta - \alpha \frac{\partial J(\theta)}{\partial \theta} \\ &= \theta + \alpha \underbrace{(V^{\pi}(s) - V_{\theta}(s))}_{\text{Prediction error}} x(s) \end{aligned}$$

Step size          Prediction error          Feature value

# Monte-Carlo with Value Function Approx.

$$\theta \leftarrow \theta + \alpha(V^\pi(s) - V_\theta(s))x(s)$$

- Now we specify the target value function  $V^\pi(s)$
- We can apply supervised learning to “training data”

$$\langle s_1, G_1 \rangle, \langle s_2, G_2 \rangle, \dots, \langle s_T, G_T \rangle$$

- For each data instance  $\langle s_t, G_t \rangle$

$$\theta \leftarrow \theta + \alpha(G_t - V_\theta(s))x(s_t)$$

- MC evaluation at least converges to a local optimum
  - In linear case it converges to a global optimum

# TD Learning with Value Function Approx.

$$\theta \leftarrow \theta + \alpha(V^\pi(s) - V_\theta(s))x(s)$$

- TD target  $r_{t+1} + \gamma V_\theta(s_{t+1})$  is a biased sample of true target value  $V^\pi(s_t)$

- Supervised learning from “training data”

$$\langle s_1, r_2 + \gamma V_\theta(s_2) \rangle, \langle s_2, r_3 + \gamma V_\theta(s_3) \rangle, \dots, \langle s_T, r_T \rangle$$

- For each data instance  $\langle s_t, r_{t+1} + \gamma V_\theta(s_{t+1}) \rangle$

$$\theta \leftarrow \theta + \alpha(r_{t+1} + \gamma V_\theta(s_{t+1}) - V_\theta(s))x(s)$$

- Linear TD converges (close) to global optimum



# Action-Value Function Approximation

- Approximate the action-value function

$$Q_{\theta}(s, a) \simeq Q^{\pi}(s, a)$$

- Minimize mean squared error

$$J(\theta) = \mathbb{E}_{\pi} \left[ \frac{1}{2} (Q^{\pi}(s, a) - Q_{\theta}(s, a))^2 \right]$$

- Stochastic gradient descent on one sample

$$\begin{aligned} \theta &\leftarrow \theta - \alpha \frac{\partial J(\theta)}{\partial \theta} \\ &= \theta + \alpha (Q^{\pi}(s, a) - Q_{\theta}(s, a)) \frac{\partial Q_{\theta}(s, a)}{\partial \theta} \end{aligned}$$

# Linear Action-Value Function Approx.

- Represent state-action pair by a feature vector

$$x(s, a) = \begin{bmatrix} x_1(s, a) \\ \vdots \\ x_k(s, a) \end{bmatrix}$$

- Parametric Q function, e.g., the linear case

$$Q_\theta(s, a) = \theta^\top x(s, a)$$

- Stochastic gradient descent update

$$\begin{aligned} \theta &\leftarrow \theta - \alpha \frac{\partial J(\theta)}{\partial \theta} \\ &= \theta + \alpha (Q^\pi(s, a) - \theta^\top x(s, a)) x(s, a) \end{aligned}$$

# TD Learning with Value Function Approx.

$$\theta \leftarrow \theta + \alpha(Q^\pi(s, a) - Q_\theta(s, a)) \frac{\partial Q_\theta(s, a)}{\partial \theta}$$

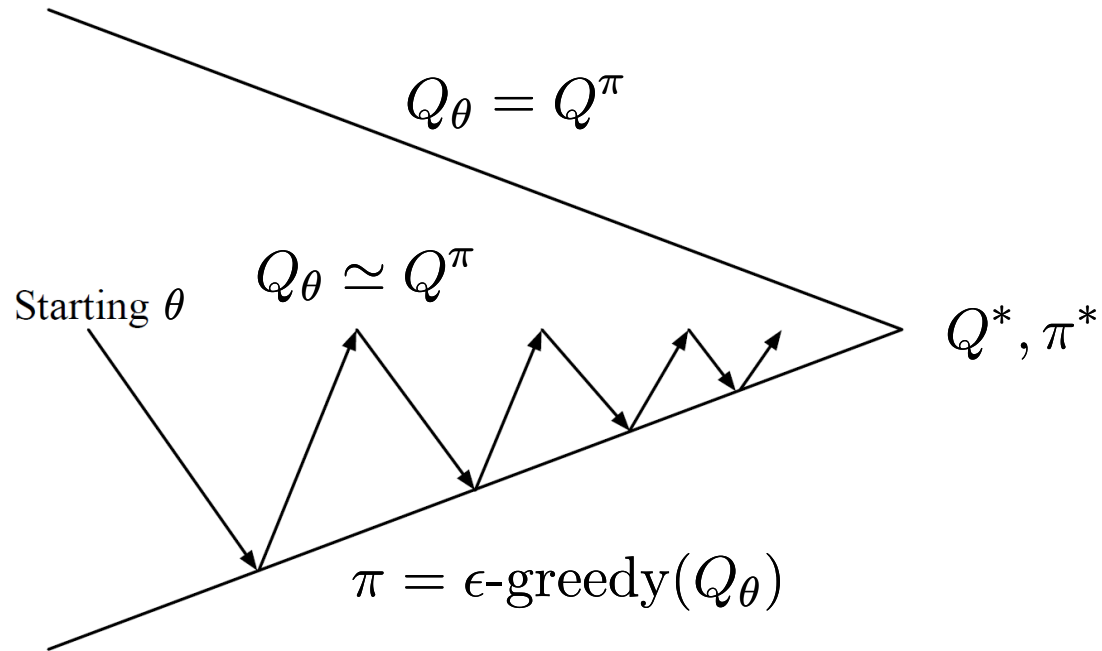
- For MC, the target is the return  $G_t$

$$\theta \leftarrow \theta + \alpha(G_t - Q_\theta(s, a)) \frac{\partial Q_\theta(s, a)}{\partial \theta}$$

- For TD, the target is  $r_{t+1} + \gamma Q_\theta(s_{t+1}, a_{t+1})$

$$\theta \leftarrow \theta + \alpha(r_{t+1} + \gamma Q_\theta(s_{t+1}, a_{t+1}) - Q_\theta(s, a)) \frac{\partial Q_\theta(s, a)}{\partial \theta}$$

# Control with Value Function Approx.



- Policy evaluation: **approximately** policy evaluation  $Q_\theta \simeq Q^\pi$
- Policy improvement:  $\epsilon$ -greedy policy improvement

# NOTE of TD Update

- For TD(0), the TD target is

- State value

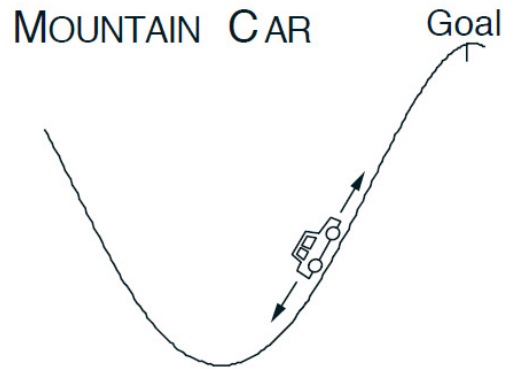
$$\begin{aligned}\theta &\leftarrow \theta + \alpha(V^\pi(s_t) - V_\theta(s_t)) \frac{\partial V_\theta(s_t)}{\partial \theta} \\ &= \theta + \alpha(r_{t+1} + \gamma V_\theta(s_{t+1}) - V_\theta(s_t)) \frac{\partial V_\theta(s_t)}{\partial \theta}\end{aligned}$$

- Action value

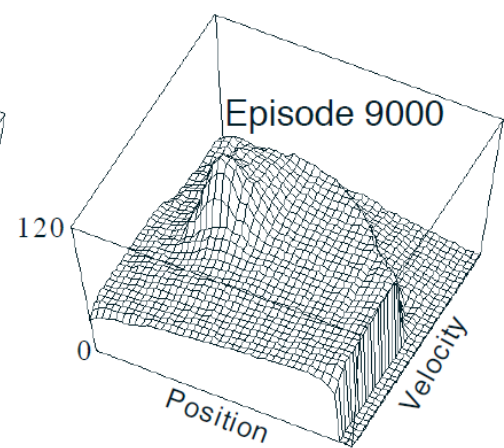
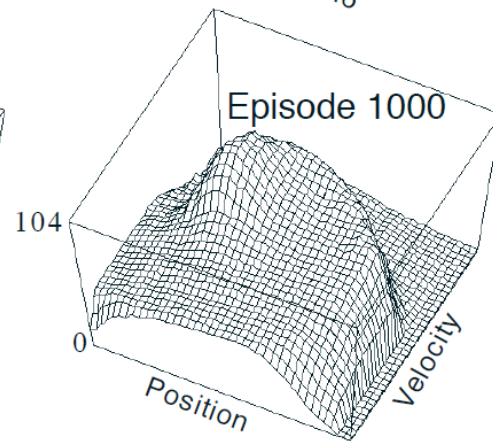
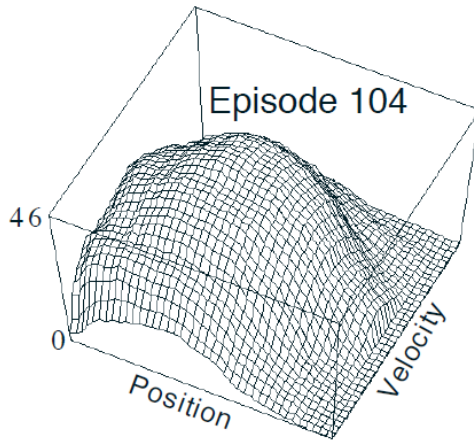
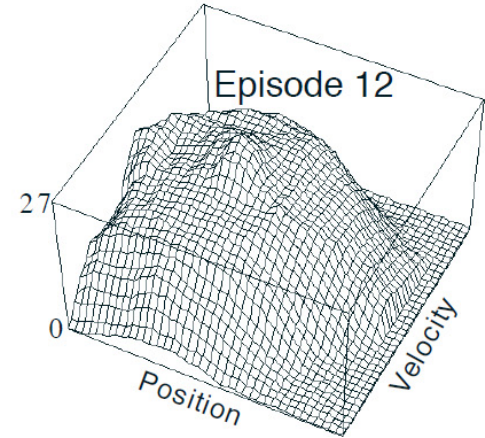
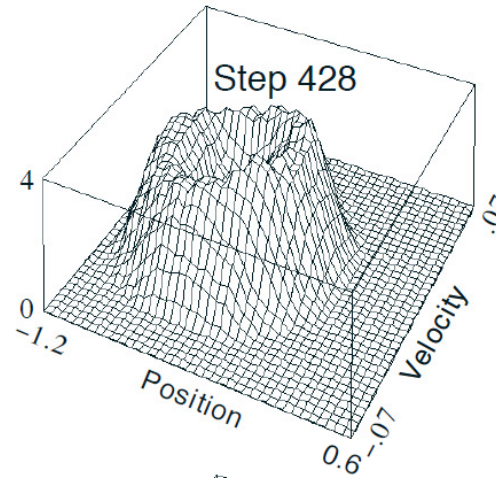
$$\begin{aligned}\theta &\leftarrow \theta + \alpha(Q^\pi(s, a) - Q_\theta(s, a)) \frac{\partial Q_\theta(s, a)}{\partial \theta} \\ &= \theta + \alpha(r_{t+1} + \gamma Q_\theta(s_{t+1}, a_{t+1}) - Q_\theta(s, a)) \frac{\partial Q_\theta(s, a)}{\partial \theta}\end{aligned}$$

- Although  $\vartheta$  is in the TD target, we don't calculate gradient from the target. Think about why.

# Case Study: Mountain Car



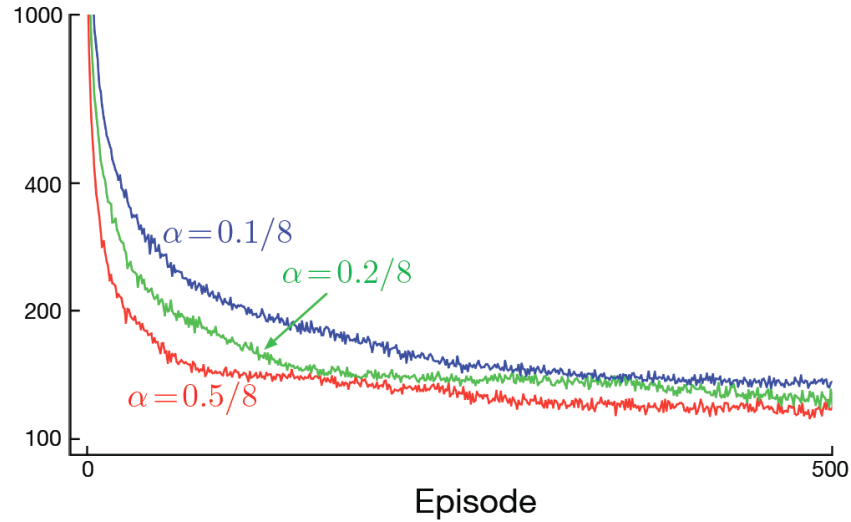
The gravity is stronger than the car's engine



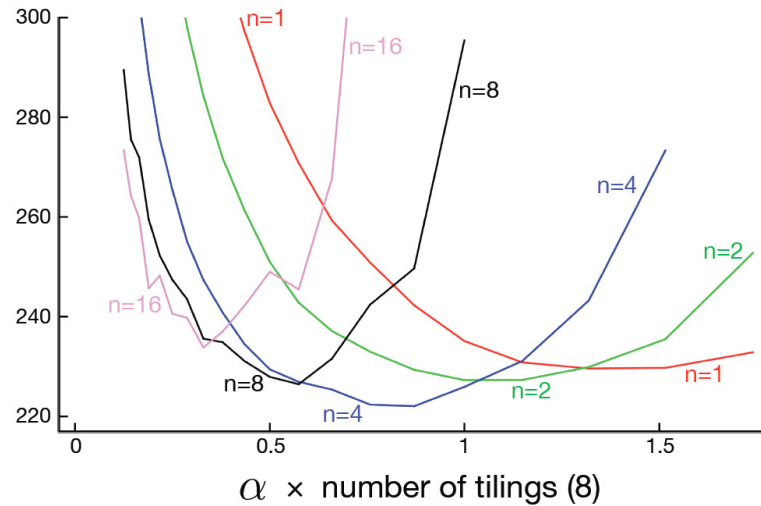
Cost-to-go function

# Case Study: Mountain Car

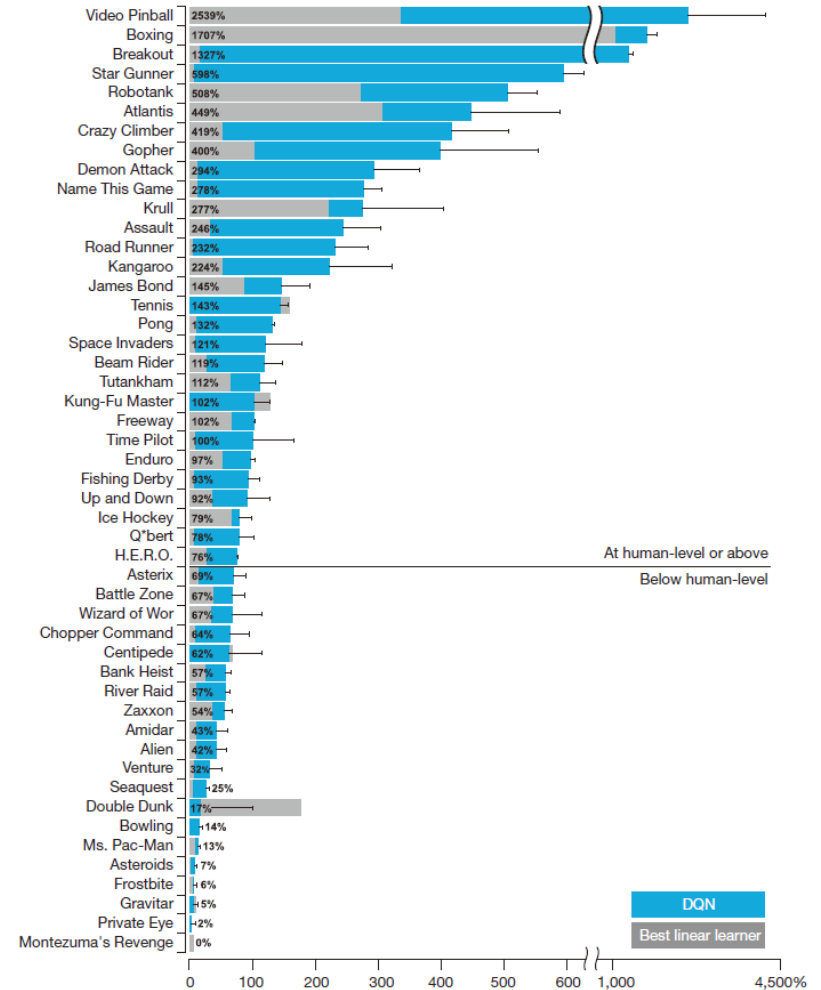
Mountain Car  
Steps per episode  
log scale  
averaged over 100 runs



Mountain Car  
Steps per episode  
averaged over  
first 50 episodes  
and 100 runs



# Deep Q-Network (DQN)



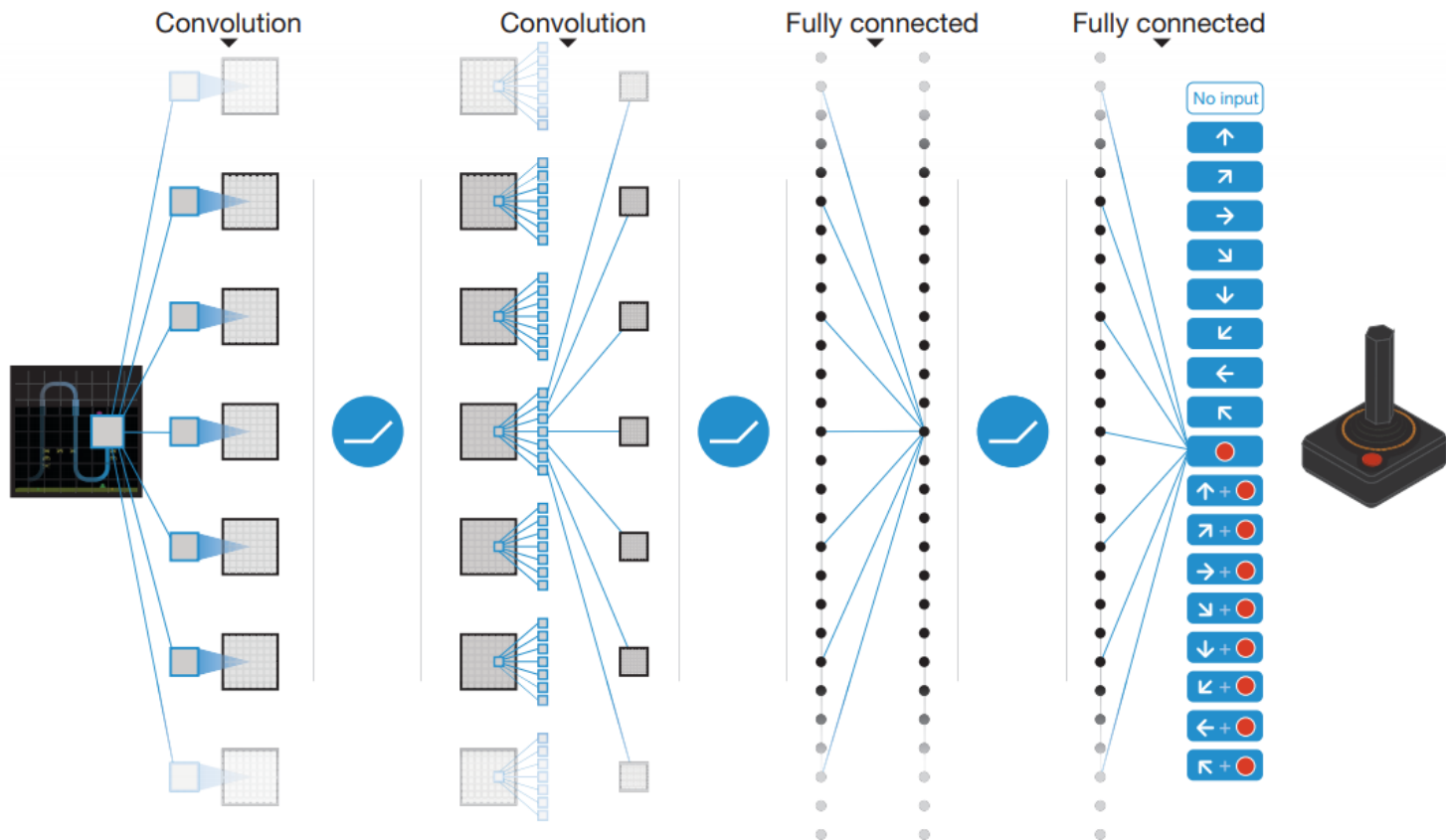
Volodymyr Mnih, Koray Kavukcuoglu, David Silver et al. Playing Atari with Deep Reinforcement Learning. NIPS 2013 workshop.

Volodymyr Mnih, Koray Kavukcuoglu, David Silver et al. Human-level control through deep reinforcement learning. Nature 2015.



# Deep Q-Network (DQN)

- Implement Q function with deep neural network



Volodymyr Mnih, Koray Kavukcuoglu, David Silver et al. Playing Atari with Deep Reinforcement Learning. NIPS 2013 workshop.

Volodymyr Mnih, Koray Kavukcuoglu, David Silver et al. Human-level control through deep reinforcement learning. Nature 2015.

# Deep Q-Network (DQN)

- The loss function of Q-learning update at iteration  $i$

$$L_i(\theta_i) = \mathbb{E}_{(s,a,r,s') \sim U(D)} \left[ \underbrace{\left( r + \gamma \max_{a'} Q(s', a'; \theta_i^-) \right)}_{\text{target Q value}} - \underbrace{Q(s, a; \theta_i)}_{\text{estimated Q value}} \right]^2$$

- $\vartheta_i$  are the network parameters to be updated at iteration  $i$ 
  - Updated with standard back-propagation algorithms
- $\vartheta_i^-$  are the target network parameters
  - Only updated with  $\vartheta_i$  for every  $C$  steps
- $(s,a,r,s') \sim U(D)$ : the samples are uniformly drawn from the experience pool  $D$ 
  - Thus to avoid the overfitting to the recent experiences

# Content

- Solutions for large MDPs
  - Discretize or bucketize states/actions
  - Build parametric value function approximation
- **Policy gradient**
- Deep reinforcement learning and multi-agent RL

# Parametric Policy

- We can parametrize the policy

$$\pi_{\theta}(a|s)$$

which could be deterministic

$$a = \pi_{\theta}(s)$$

or stochastic

$$\pi_{\theta}(a|s) = P(a|s; \theta)$$

- $\vartheta$  is the parameters of the policy
- Generalize from seen states to unseen states
- We focus on model-free reinforcement learning

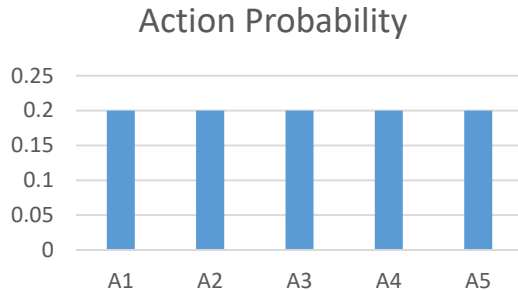
# Policy-based RL

- Advantages
  - Better convergence properties
  - Effective in high-dimensional or continuous action spaces
    - No.1 reason: for value function, you have to take a max operation
  - Can learn stochastic policies
- Disadvantages
  - Typically converge to a local rather than global optimum
  - Evaluating a policy is typically inefficient and of high variance

# Policy Gradient

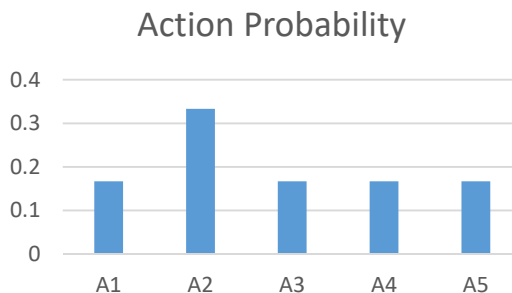
- For stochastic policy  $\pi_{\theta}(a|s) = P(a|s; \theta)$
- Intuition
  - lower the probability of the action that leads to low value/reward
  - higher the probability of the action that leads to high value/reward
- A 5-action example

1. Initialize  $\vartheta$



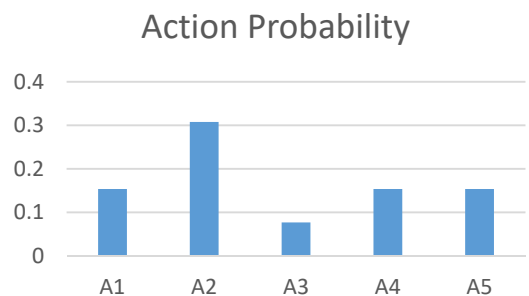
2. Take action A2  
Observe positive reward

3. Update  $\vartheta$  by policy gradient



4. Take action A3  
Observe negative reward

5. Update  $\vartheta$  by policy gradient



# Policy Gradient in One-Step MDPs

- Consider a simple class of one-step MDPs
  - Starting in state  $s \sim d(s)$
  - Terminating after one time-step with reward  $r_{sa}$
- Policy expected value

$$J(\theta) = \mathbb{E}_{\pi_{\theta}}[r] = \sum_{s \in \mathcal{S}} d(s) \sum_{a \in \mathcal{A}} \pi_{\theta}(a|s) r_{sa}$$

$$\frac{\partial J(\theta)}{\partial \theta} = \sum_{s \in \mathcal{S}} d(s) \sum_{a \in \mathcal{A}} \frac{\partial \pi_{\theta}(a|s)}{\partial \theta} r_{sa}$$

# Likelihood Ratio

- Likelihood ratios exploit the following identity

$$\begin{aligned}\frac{\partial \pi_{\theta}(a|s)}{\partial \theta} &= \pi_{\theta}(a|s) \frac{1}{\pi_{\theta}(a|s)} \frac{\partial \pi_{\theta}(a|s)}{\partial \theta} \\ &= \pi_{\theta}(a|s) \frac{\partial \log \pi_{\theta}(a|s)}{\partial \theta}\end{aligned}$$

- Thus the policy's expected value

$$\begin{aligned}J(\theta) &= \mathbb{E}_{\pi_{\theta}}[r] = \sum_{s \in \mathcal{S}} d(s) \sum_{a \in \mathcal{A}} \pi_{\theta}(a|s) r_{sa} \\ \frac{\partial J(\theta)}{\partial \theta} &= \sum_{s \in \mathcal{S}} d(s) \sum_{a \in \mathcal{A}} \frac{\partial \pi_{\theta}(a|s)}{\partial \theta} r_{sa} \\ &= \sum_{s \in \mathcal{S}} d(s) \sum_{a \in \mathcal{A}} \pi_{\theta}(a|s) \frac{\partial \log \pi_{\theta}(a|s)}{\partial \theta} r_{sa} \\ &= \mathbb{E}_{\pi_{\theta}} \left[ \frac{\partial \log \pi_{\theta}(a|s)}{\partial \theta} r_{sa} \right] \quad \text{This can be approximated by sampling state } s \text{ from } d(s) \text{ and action } a \text{ from } \pi_{\theta}\end{aligned}$$



# Policy Gradient Theorem

- The policy gradient theorem generalizes the likelihood ratio approach to multi-step MDPs
  - Replaces instantaneous reward  $r_{sa}$  with long-term value  $Q^{\pi_\theta}(s, a)$
- Policy gradient theorem applies to
  - start state objective  $J_1$ , average reward objective  $J_{avR}$ , and average value objective  $J_{avV}$
- Theorem
  - For any differentiable policy  $\pi_\theta(a|s)$ , for any of policy objective function  $J = J_1, J_{avR}, J_{avV}$ , the policy gradient is

$$\frac{\partial J(\theta)}{\partial \theta} = \mathbb{E}_{\pi_\theta} \left[ \frac{\partial \log \pi_\theta(a|s)}{\partial \theta} Q^{\pi_\theta}(s, a) \right]$$

Please refer to appendix of the slides for detailed proofs

# Monte-Carlo Policy Gradient (REINFORCE)

- Update parameters by stochastic gradient ascent
- Using policy gradient theorem
- Using return  $G_t$  as an unbiased sample of  $Q^{\pi_\theta}(s, a)$

$$\Delta\theta_t = \alpha \frac{\partial \log \pi_\theta(a_t|s_t)}{\partial \theta} G_t$$

- REINFORCE Algorithm

Initialize  $\vartheta$  arbitrarily

for each episode  $\{s_1, a_1, r_2, \dots, s_{T-1}, a_{T-1}, r_T\} \sim \pi_\theta$  do

for  $t=1$  to  $T-1$  do

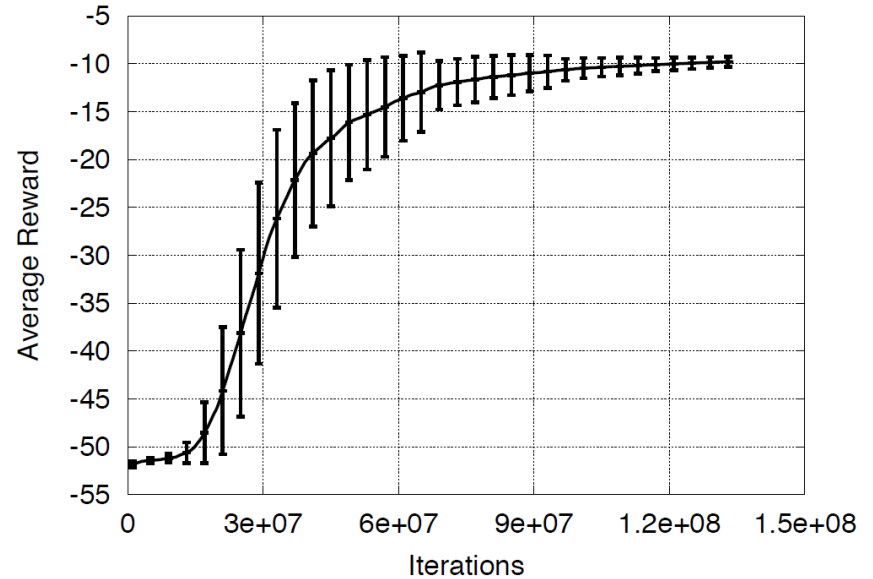
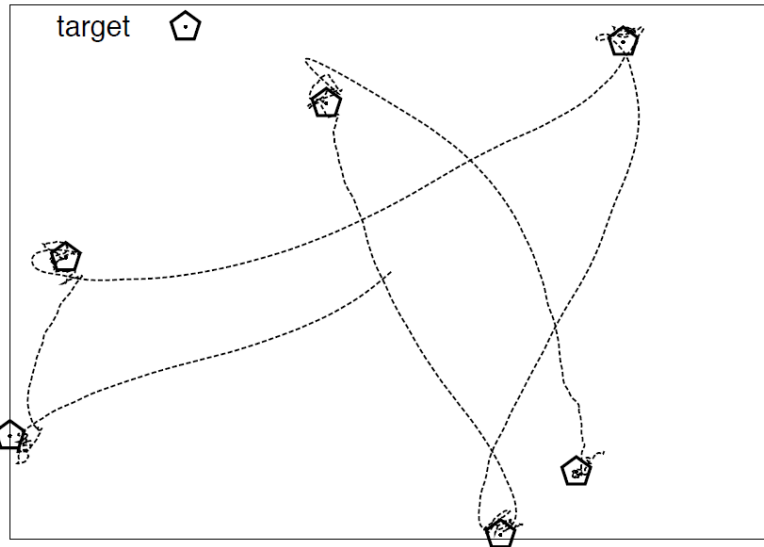
$$\theta \leftarrow \theta + \alpha \frac{\partial}{\partial \theta} \log \pi_\theta(a_t|s_t) G_t$$

end for

end for

return  $\vartheta$

# Puck World Example



- Continuous actions exert small force on puck
- Puck is rewarded for getting close to target
- Target location is reset every 30 seconds
- Policy is trained using variant of MC policy gradient

# Softmax Stochastic Policy

- Softmax policy is a very commonly used stochastic policy

$$\pi_{\theta}(a|s) = \frac{e^{f_{\theta}(s,a)}}{\sum_{a'} e^{f_{\theta}(s,a')}}$$

- where  $f_{\vartheta}(s,a)$  is the score function of a state-action pair parametrized by  $\vartheta$ , which can be defined with domain knowledge
- The gradient of its log-likelihood

$$\begin{aligned} \frac{\partial \log \pi_{\theta}(a|s)}{\partial \theta} &= \frac{\partial f_{\theta}(s, a)}{\partial \theta} - \frac{1}{\sum_{a'} e^{f_{\theta}(s, a')}} \sum_{a''} e^{f_{\theta}(s, a'')} \frac{\partial f_{\theta}(s, a'')}{\partial \theta} \\ &= \frac{\partial f_{\theta}(s, a)}{\partial \theta} - \mathbb{E}_{a' \sim \pi_{\theta}(a'|s)} \left[ \frac{\partial f_{\theta}(s, a')}{\partial \theta} \right] \end{aligned}$$

# Softmax Stochastic Policy

- Softmax policy is a very commonly used stochastic policy

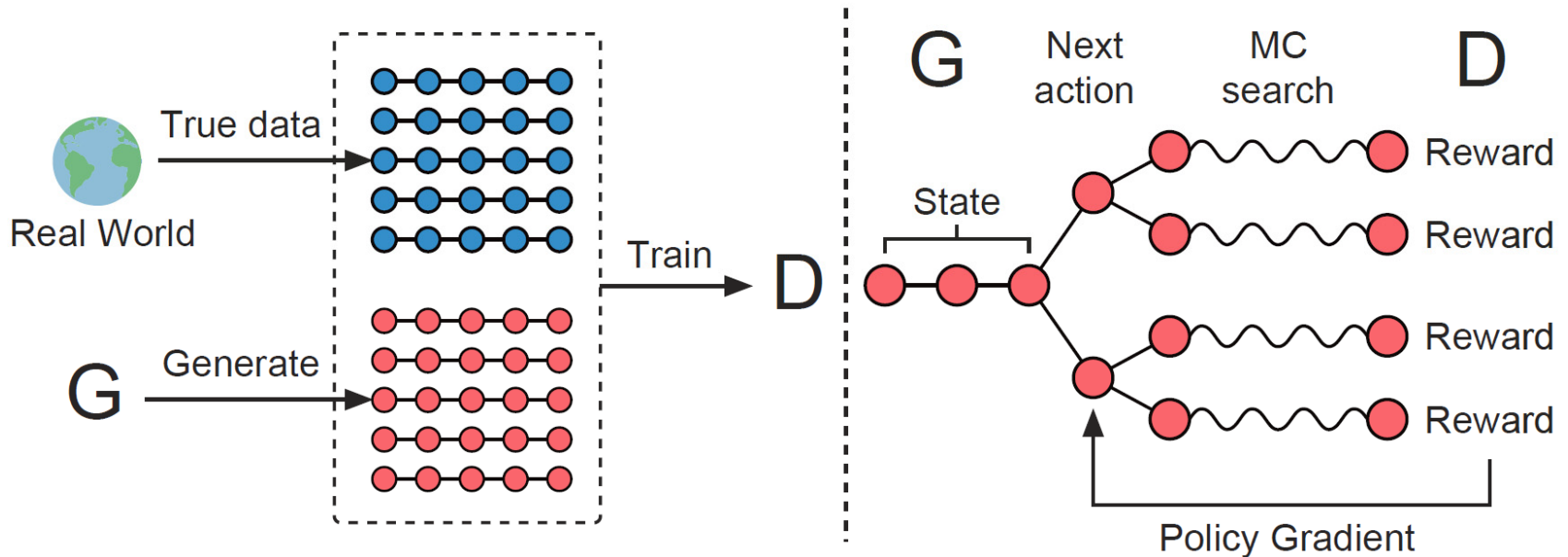
$$\pi_{\theta}(a|s) = \frac{e^{f_{\theta}(s,a)}}{\sum_{a'} e^{f_{\theta}(s,a')}}$$

- where  $f_{\vartheta}(s,a)$  is the score function of a state-action pair parametrized by  $\vartheta$ , which can be defined with domain knowledge
- For example, we define the linear score function

$$f_{\theta}(s, a) = \theta^{\top} x(s, a)$$

$$\begin{aligned} \frac{\partial \log \pi_{\theta}(a|s)}{\partial \theta} &= \frac{\partial f_{\theta}(s, a)}{\partial \theta} - \mathbb{E}_{a' \sim \pi_{\theta}(a'|s)} \left[ \frac{\partial f_{\theta}(s, a')}{\partial \theta} \right] \\ &= x(s, a) - \mathbb{E}_{a' \sim \pi_{\theta}(a'|s)} [x(s, a')] \end{aligned}$$

# Sequence Generation Example



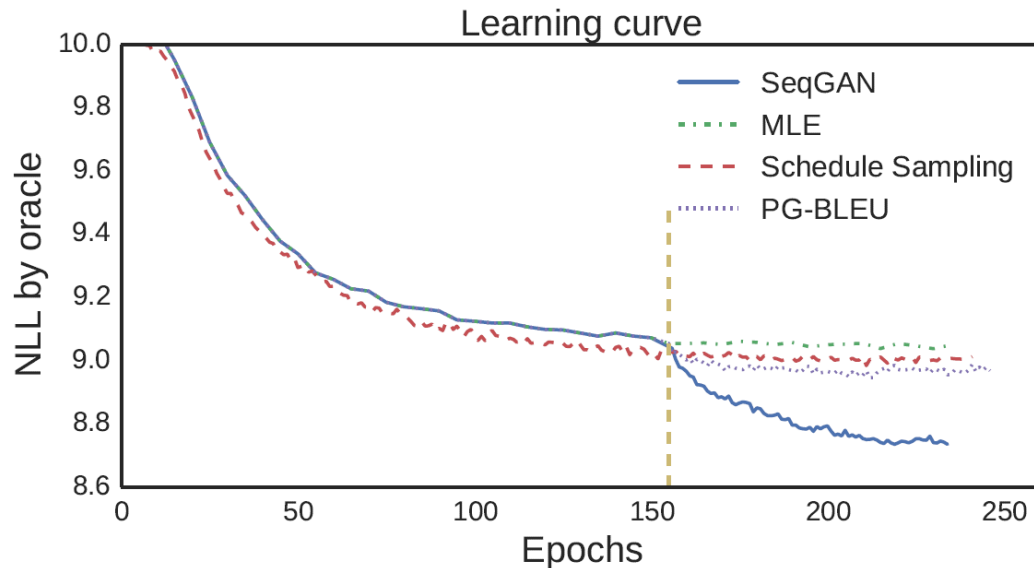
- Generator is a reinforcement learning policy  $G_{\theta}(y_t|Y_{1:t-1})$  of generating a sequence
  - Decide the next word (discrete action) to generate given the previous ones, implemented by softmax policy
  - Discriminator provides the reward (i.e. the probability of being true data) for the whole sequence
  - $G$  is trained by MC policy gradient (REINFORCE)

# Experiments on Synthetic Data

- Evaluation measure with Oracle

$$\text{NLL}_{\text{oracle}} = -\mathbb{E}_{Y_{1:T} \sim G_{\theta}} \left[ \sum_{t=1}^T \log G_{\text{oracle}}(y_t | Y_{1:t-1}) \right]$$

Algorithm	Random	MLE	SS	PG-BLEU	SeqGAN
NLL	10.310	9.038	8.985	8.946	<b>8.736</b>
<i>p</i> -value	$< 10^{-6}$	$< 10^{-6}$	$< 10^{-6}$	$< 10^{-6}$	



# Content

- Solutions for large MDPs
  - Discretize or bucketize states/actions
  - Build parametric value function approximation
- Policy gradient
- Deep reinforcement learning and multi-agent RL
  - By our invited speakers Yuhuai Wu, Shixiang Gu and Ying Wen



# Policy Gradient Theorem: Average Reward Setting

- Average reward objective

$$\begin{aligned}
 J(\pi) &= \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \left[ r_1 + r_2 + \dots + r_n | \pi \right] = \sum_s d^\pi(s) \sum_a \pi(s, a) r(s, a) \\
 Q^\pi(s, a) &= \sum_{t=1}^{\infty} \mathbb{E} \left[ r_t - J(\pi) | s_0 = s, a_0 = a, \pi \right] \\
 \frac{\partial V^\pi(s)}{\partial \theta} &\stackrel{\text{def}}{=} \frac{\partial}{\partial \theta} \sum_a \pi(s, a) Q^\pi(s, a), \quad \forall s \\
 &= \sum_a \left[ \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a) + \pi(s, a) \frac{\partial}{\partial \theta} Q^\pi(s, a) \right] \\
 &= \sum_a \left[ \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a) + \pi(s, a) \frac{\partial}{\partial \theta} \left( r(s, a) - J(\pi) + \sum_{s'} P_{ss'}^a V^\pi(s') \right) \right] \\
 &= \sum_a \left[ \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a) + \pi(s, a) \left( -\frac{\partial J(\pi)}{\partial \theta} + \frac{\partial}{\partial \theta} \sum_{s'} P_{ss'}^a V^\pi(s') \right) \right] \\
 \Rightarrow \frac{\partial J(\pi)}{\partial \theta} &= \sum_a \left[ \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a) + \pi(s, a) \sum_{s'} P_{ss'}^a \frac{\partial V^\pi(s')}{\partial \theta} \right] - \frac{\partial V^\pi(s)}{\partial \theta}
 \end{aligned}$$

# Policy Gradient Theorem: Average Reward Setting

- Average reward objective

$$\begin{aligned}
 \frac{\partial J(\pi)}{\partial \theta} &= \sum_a \left[ \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a) + \pi(s, a) \sum_{s'} P_{ss'}^a \frac{\partial V^\pi(s')}{\partial \theta} \right] - \frac{\partial V^\pi(s)}{\partial \theta} \\
 \sum_s d^\pi(s) \frac{\partial J(\pi)}{\partial \theta} &= \sum_s d^\pi(s) \sum_a \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a) + \sum_s d^\pi(s) \sum_a \pi(s, a) \sum_{s'} P_{ss'}^a \frac{\partial V^\pi(s')}{\partial \theta} - \sum_s d^\pi(s) \frac{\partial V^\pi(s)}{\partial \theta} \\
 \sum_s d^\pi(s) \sum_a \pi(s, a) \sum_{s'} P_{ss'}^a \frac{\partial V^\pi(s')}{\partial \theta} &= \sum_s \sum_a \sum_{s'} d^\pi(s) \pi(s, a) P_{ss'}^a \frac{\partial V^\pi(s')}{\partial \theta} \\
 &= \sum_s \sum_{s'} d^\pi(s) \left( \sum_a \pi(s, a) P_{ss'}^a \right) \frac{\partial V^\pi(s')}{\partial \theta} = \sum_s \sum_{s'} d^\pi(s) P_{ss'} \frac{\partial V^\pi(s')}{\partial \theta} \\
 &= \sum_{s'} \left( \sum_s d^\pi(s) P_{ss'} \right) \frac{\partial V^\pi(s')}{\partial \theta} = \sum_{s'} d^\pi(s') \frac{\partial V^\pi(s')}{\partial \theta} \\
 \Rightarrow \sum_s d^\pi(s) \frac{\partial J(\pi)}{\partial \theta} &= \sum_s d^\pi(s) \sum_a \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a) + \sum_{s'} d^\pi(s') \frac{\partial V^\pi(s')}{\partial \theta} - \sum_s d^\pi(s) \frac{\partial V^\pi(s)}{\partial \theta} \\
 \Rightarrow \frac{\partial J(\pi)}{\partial \theta} &= \sum_s d^\pi(s) \sum_a \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a)
 \end{aligned}$$

# Policy gradient theorem: Start Value Setting

- Start state value objective

$$\begin{aligned}
 J(\pi) &= \mathbb{E} \left[ \sum_{t=1}^{\infty} \gamma^{t-1} r_t \mid s_0, \pi \right] \\
 Q^\pi(s, a) &= \mathbb{E} \left[ \sum_{k=1}^{\infty} \gamma^{k-1} r_{t+k} \mid s_t = s, a_t = a, \pi \right] \\
 \frac{\partial V^\pi(s)}{\partial \theta} &\stackrel{\text{def}}{=} \frac{\partial}{\partial \theta} \sum_a \pi(s, a) Q^\pi(s, a), \quad \forall s \\
 &= \sum_a \left[ \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a) + \pi(s, a) \frac{\partial}{\partial \theta} Q^\pi(s, a) \right] \\
 &= \sum_a \left[ \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a) + \pi(s, a) \frac{\partial}{\partial \theta} \left( r(s, a) + \sum_{s'} \gamma P_{ss'}^a V^\pi(s') \right) \right] \\
 &= \sum_a \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a) + \sum_a \pi(s, a) \gamma \sum_{s'} P_{ss'}^a \frac{\partial V^\pi(s')}{\partial \theta}
 \end{aligned}$$

# Policy gradient theorem: Start Value Setting

- Start state value objective

$$\frac{\partial V^\pi(s)}{\partial \theta} = \sum_a \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a) + \sum_a \pi(s, a) \gamma \sum_{s_1} P_{ss_1}^a \frac{\partial V^\pi(s_1)}{\partial \theta}$$

$$\sum_a \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a) = \gamma^0 Pr(s \rightarrow s, 0, \pi) \sum_a \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a)$$

$$\sum_a \pi(s, a) \gamma \sum_{s_1} P_{ss_1}^a \frac{\partial V^\pi(s_1)}{\partial \theta} = \sum_{s_1} \sum_a \pi(s, a) \gamma P_{ss_1}^a \frac{\partial V^\pi(s_1)}{\partial \theta}$$

$$= \sum_{s_1} \gamma P_{ss_1} \frac{\partial V^\pi(s_1)}{\partial \theta} = \gamma^1 \sum_{s_1} Pr(s \rightarrow s_1, 1, \pi) \frac{\partial V^\pi(s_1)}{\partial \theta}$$

$$\frac{\partial V^\pi(s_1)}{\partial \theta} = \sum_a \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a) + \gamma^1 \sum_{s_2} Pr(s_1 \rightarrow s_2, 1, \pi) \frac{\partial V^\pi(s_2)}{\partial \theta}$$

## APPENDIX

# Policy gradient theorem: Start Value Setting

- Start state value objective

$$\begin{aligned}
 \frac{\partial V^\pi(s)}{\partial \theta} &= \gamma^0 Pr(s \rightarrow s, 0, \pi) \sum_a \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a) + \gamma^1 \sum_{s_1} Pr(s \rightarrow s_1, 1, \pi) \sum_a \frac{\partial \pi(s_1, a)}{\partial \theta} Q^\pi(s_1, a) \\
 &\quad + \gamma^2 \sum_{s_1} Pr(s \rightarrow s_1, 1, \pi) \sum_{s_2} Pr(s_1 \rightarrow s_2, 1, \pi) \frac{\partial V^\pi(s_2)}{\partial \theta} \\
 &= \gamma^0 Pr(s \rightarrow s, 0, \pi) \sum_a \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a) + \gamma^1 \sum_{s_1} Pr(s \rightarrow s_1, 1, \pi) \sum_a \frac{\partial \pi(s_1, a)}{\partial \theta} Q^\pi(s_1, a) \\
 &\quad + \gamma^2 \sum_{s_2} Pr(s \rightarrow s_2, 2, \pi) \frac{\partial V^\pi(s_2)}{\partial \theta} \\
 &= \sum_{k=0}^{\infty} \sum_x \gamma^k Pr(s \rightarrow x, k, \pi) \sum_a \frac{\partial \pi(x, a)}{\partial \theta} Q^\pi(x, a) = \sum_x \sum_{k=0}^{\infty} \gamma^k Pr(s \rightarrow x, k, \pi) \sum_a \frac{\partial \pi(x, a)}{\partial \theta} Q^\pi(x, a) \\
 \Rightarrow \frac{\partial J(\pi)}{\partial \theta} &= \frac{\partial V^\pi(s_0)}{\partial \theta} = \sum_s \sum_{k=0}^{\infty} \gamma^k Pr(s_0 \rightarrow s, k, \pi) \sum_a \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a) = \sum_s d^\pi(s) \sum_a \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a)
 \end{aligned}$$