CS420, Machine Learning, Lecture 12

Approximation Methods in Reinforcement Learning

Weinan Zhang

Shanghai Jiao Tong University

http://wnzhang.net

http://wnzhang.net/teaching/cs420/index.html

Reinforcement Learning Materials

Our course on RL is mainly based on the materials from these masters.

Prof. Richard Sutton

- University of Alberta, Canada
- http://incompleteideas.net/sutton/index.html
- Reinforcement Learning: An Introduction (2nd edition)
- http://incompleteideas.net/sutton/book/the-book-2nd.html

Dr. David Silver

- Google DeepMind and UCL, UK
- http://www0.cs.ucl.ac.uk/staff/d.silver/web/Home.html
- UCL Reinforcement Learning Course
- http://www0.cs.ucl.ac.uk/staff/d.silver/web/Teaching.html

Prof. Andrew Ng

- Stanford University, US
- http://www.andrewng.org/
- Machine Learning (CS229) Lecture Notes 12: RL
- http://cs229.stanford.edu/materials.html

Last Lecture

- Model-based dynamic programming
  - Value iteration $V(s) = R(s) + \max_{a \in A} \gamma \sum_{s' \in S} P_{sa}(s')V(s')$
  - Policy iteration $\pi(s) = \arg\max_{a \in A} \sum_{s' \in S} P_{sa}(s')V(s')$
- Model-free reinforcement learning
  - On-policy MC $V(s_t) \leftarrow V(s_t) + \alpha(G_t - V(s_t))$
  - On-policy TD $V(s_t) \leftarrow V(s_t) + \alpha(r_{t+1} + \gamma V(s_{t+1}) - V(s_t))$
  - On-policy TD SARSA

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_{t+1} + \gamma Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t))$$

  - Off-policy TD Q-learning

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_{t+1} + \gamma \max_{a'} Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t))$$

Key Problem to Solve in This Lecture

- In all previous models, we have created a lookup table to maintain a variable $V(s)$ for each state or $Q(s,a)$ for each state-action

- What if we have a large MDP, i.e.
  - the state or state-action space is too large
  - or the state or action space is continuous

  to maintain $V(s)$ for each state or $Q(s,a)$ for each state-action?
  - For example
    - Game of Go (10^{170} states)
    - Helicopter, autonomous car (continuous state space)

Content

- Solutions for large MDPs
  - Discretize or bucketize states/actions
  - Build parametric value function approximation

- Policy gradient

- Deep reinforcement learning

Content

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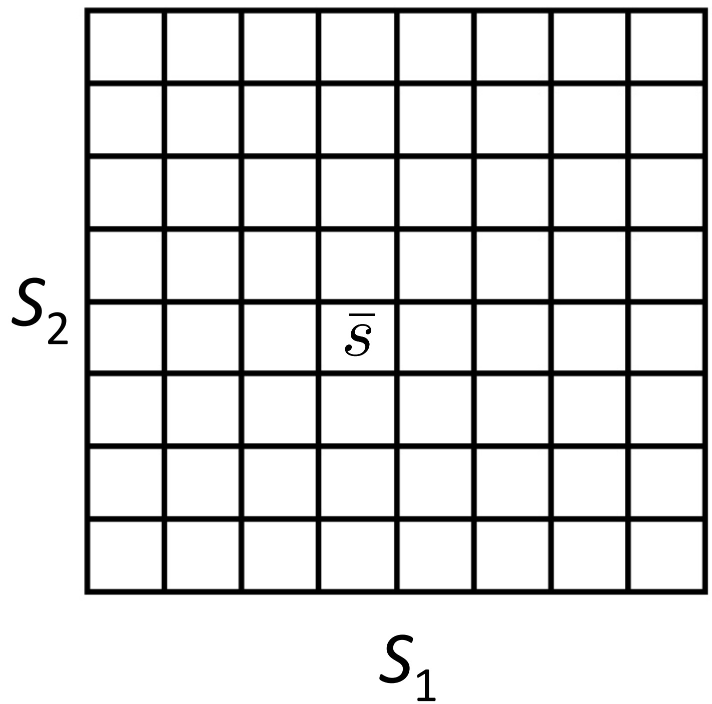
Discretization Continuous MDP

- For a continuous-state MDP, we can discretize the state space
  - For example, if we have 2D states (s_1, s_2), we can use a grid to discretize the state space
  - The discrete state $\bar{s}$
  - The discretized MDP:

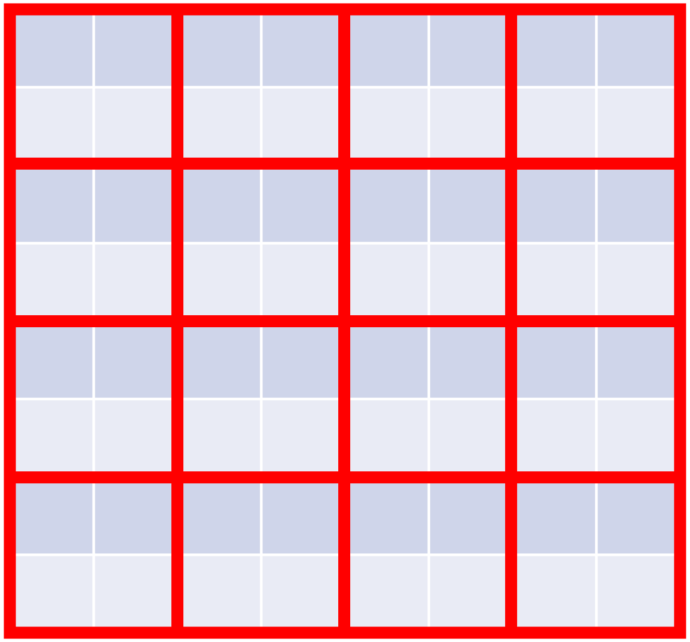
$$(\bar{S}, A, \{P_{\bar{s}a}\}, \gamma, R)$$

  - Then solve this MDP with any previous solutions



Bucketize Large Discrete MDP

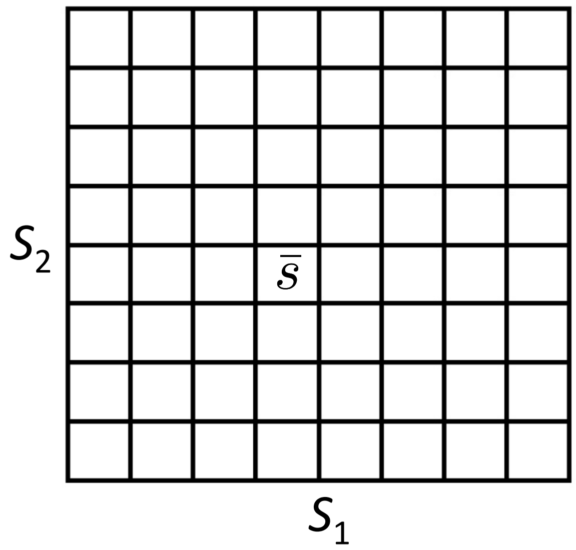
- For a large discrete-state MDP, we can bucketize the states to 'down sample' the states
  - To use domain knowledge to merge similar discrete states
    - For example, clustering using state features extracted from domain knowledge

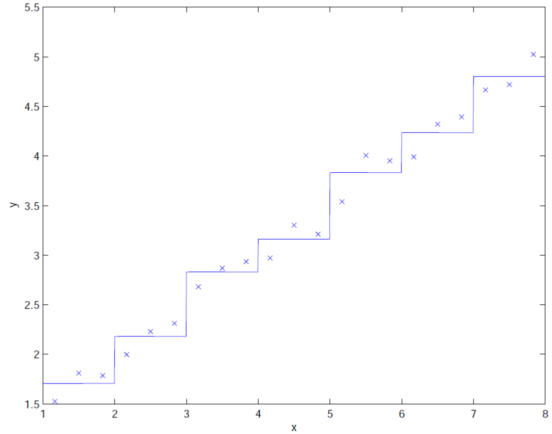


Discretization/Bucketization

- Pros
  - Straightforward and off-the-shelf
  - Efficient
  - Can work well for many problems
- Cons
  - A fairly naïve representation for *V*
  - Assumes a constant value over each discretized cell
  - Curse of dimensionality

$$S = \mathbb{R}^n \Rightarrow \bar{S} = \{1, \ldots, k\}^n$$





Parametric Value Function Approximation

- Create parametric (thus learnable) functions to approximate the value function

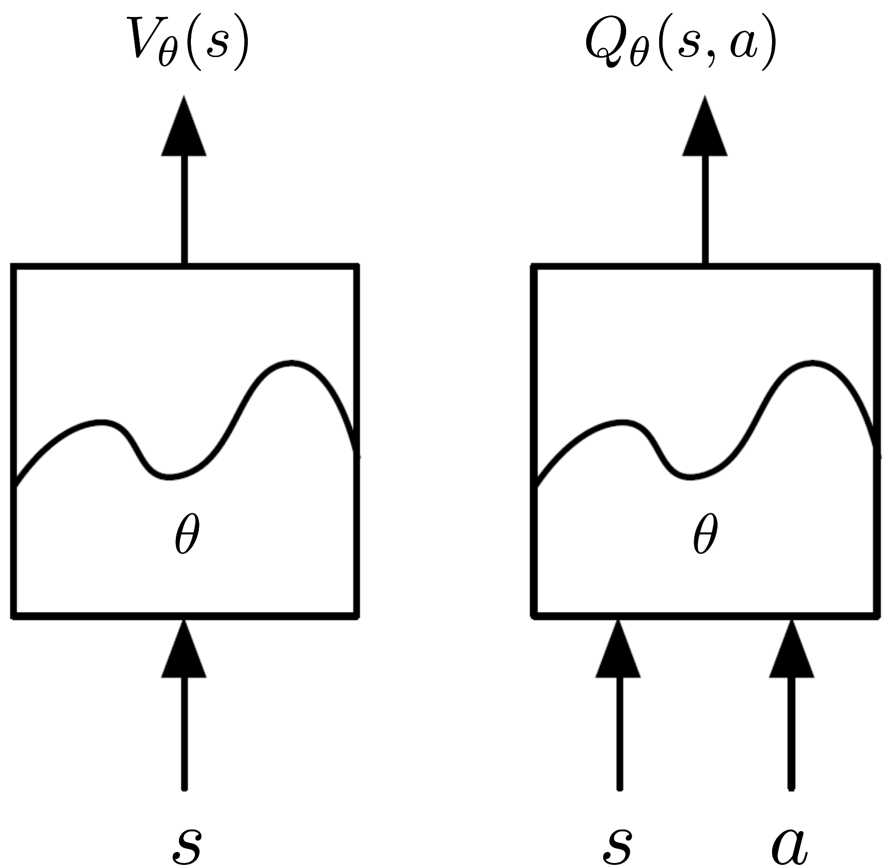
$$V_\theta(s) \simeq V^\pi(s)$$

$$Q_\theta(s, a) \simeq Q^\pi(s, a)$$

  - ϑ is the parameters of the approximation function, which can be updated by reinforcement learning

  - Generalize from seen states to unseen states

Main Types of Value Function Approx.



Many function approximations

- (Generalized) linear model
- Neural network
- Decision tree
- Nearest neighbor
- Fourier / wavelet bases

Differentiable functions

- (Generalized) linear model
- Neural network

We assume the model is suitable to be trained for non-stationary, non-iid data

Value Function Approx. by SGD

- Goal: find parameter vector ϑ minimizing mean-squared error between approximate value function $V_{\vartheta}(s)$ and true value $V^{\pi}(s)$

$$J(\theta) = \mathbb{E}_{\pi}\Big[\frac{1}{2}(V^{\pi}(s) - V_{\theta}(s))^2\Big]$$

- Gradient to minimize the error

$$-\frac{\partial J(\theta)}{\partial \theta} = \mathbb{E}_{\pi}[V^{\pi}(s) - V_{\theta}(s)]\frac{\partial V_{\theta}(s)}{\partial \theta}$$

- Stochastic gradient descent on one sample

$$\begin{aligned} \theta &\leftarrow \theta - \alpha\frac{\partial J(\theta)}{\partial \theta} \\ &= \theta + \alpha(V^{\pi}(s) - V_{\theta}(s))\frac{\partial V_{\theta}(s)}{\partial \theta} \end{aligned}$$

Featurize the State

- Represent state by a feature vector

$$x(s) = \begin{bmatrix} x_1(s) \\ \vdots \\ x_k(s) \end{bmatrix}$$

- For example of a helicopter
  - 3D location
  - 3D speed (differentiation of location)
  - 3D acceleration (differentiation of speed)

Linear Value Function Approximation

- Represent value function by a linear combination of features

$$V_\theta(s) = \theta^\top x(s)$$

- Objective function is quadratic in parameters ϑ

$$J(\theta) = \mathbb{E}_\pi\left[\frac{1}{2}\left(V^\pi(s) - \theta^\top x(s)\right)^2\right]$$

- Thus stochastic gradient descent converges on global optimum

$$\begin{aligned}\theta &\leftarrow \theta - \alpha \frac{\partial J(\theta)}{\partial \theta} \\ &= \theta + \underbrace{\alpha}_{\text{Step size}}\big(\underbrace{V^\pi(s) - V_\theta(s)}_{\text{Prediction error}}\big)\underbrace{x(s)}_{\text{Feature value}}\end{aligned}$$

Monte-Carlo with Value Function Approx.

$$\theta \leftarrow \theta + \alpha(V^\pi(s) - V_\theta(s))x(s)$$

- Now we specify the target value function $V^\pi(s)$
- We can apply supervised learning to "training data"

$$\langle s_1, G_1 \rangle, \langle s_2, G_2 \rangle, \ldots, \langle s_T, G_T \rangle$$

- For each data instance $\langle s_t, G_t \rangle$

$$\theta \leftarrow \theta + \alpha(G_t - V_\theta(s))x(s_t)$$

- MC evaluation converges to a local optimum

TD Learning with Value Function Approx.

$$\theta \leftarrow \theta + \alpha(V^\pi(s) - V_\theta(s))x(s)$$

- TD target $r_{t+1} + \gamma V_\theta(s_{t+1})$ is a biased sample of true target value $V^\pi(s_t)$
- Supervised learning from "training data"

$$\langle s_1, r_2 + \gamma V_\theta(s_2)\rangle, \langle s_2, r_3 + \gamma V_\theta(s_3)\rangle, \ldots, \langle s_T, R_T\rangle$$

- For each data instance $\langle s_t, r_{t+1} + \gamma V_\theta(s_{t+1})\rangle$

$$\theta \leftarrow \theta + \alpha(r_{t+1} + \gamma V_\theta(s_{t+1}) - V_\theta(s))x(s_t)$$

- Linear TD(0) converges (close) to global optimum [why?]

Action-Value Function Approximation

- Approximate the action-value function

$$Q_\theta(s,a) \simeq Q^\pi(s,a)$$

- Minimize mean squared error

$$J(\theta) = \mathbb{E}_\pi\Big[\frac{1}{2}(Q^\pi(s,a) - Q_\theta(s,a))^2\Big]$$

- Stochastic gradient descent on one sample

$$\begin{aligned}\theta &\leftarrow \theta - \alpha\frac{\partial J(\theta)}{\partial\theta}\\ &= \theta + \alpha(Q^\pi(s,a) - Q_\theta(s,a))\frac{\partial Q_\theta(s,a)}{\partial\theta}\end{aligned}$$

Linear Action-Value Function Approx.

- Represent state-action pair by a feature vector

$$x(s,a) = \begin{bmatrix} x_1(s,a) \\ \vdots \\ x_k(s,a) \end{bmatrix}$$

- Parametric Q function

$$Q_\theta(s,a) = \theta^\top x(s,a)$$

- Stochastic gradient descent update

$$\begin{aligned} \theta &\leftarrow \theta - \alpha \frac{\partial J(\theta)}{\partial \theta} \\ &= \theta + \alpha(Q^\pi(s,a) - \theta^\top x(s,a))x(s,a) \end{aligned}$$

TD Learning with Value Function Approx.

$$\theta \leftarrow \theta + \alpha(Q^{\pi}(s,a) - Q_{\theta}(s,a))\frac{\partial Q_{\theta}(s,a)}{\partial \theta}$$

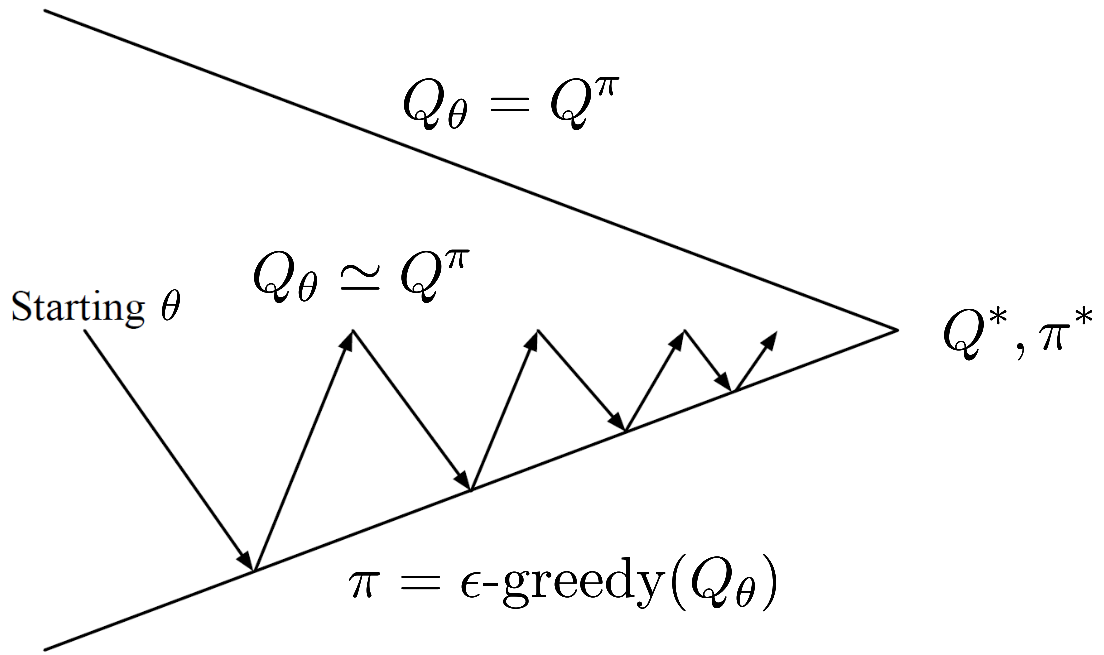
- For MC, the target is the return G_t

$$\theta \leftarrow \theta + \alpha(G_t - Q_{\theta}(s,a))\frac{\partial Q_{\theta}(s,a)}{\partial \theta}$$

- For TD(0), the TD target is $r_{t+1} + \gamma Q_{\theta}(s_{t+1}, a_{t+1})$

$$\theta \leftarrow \theta + \alpha(r_{t+1} + \gamma Q_{\theta}(s_{t+1}, a_{t+1}) - Q_{\theta}(s,a))\frac{\partial Q_{\theta}(s,a)}{\partial \theta}$$

Control with Value Function Approx.



- Policy evaluation: approximately policy evaluation $Q_\theta \simeq Q^\pi$
- Policy improvement: ε-greedy policy improvement

NOTE of TD Update

- For TD(0), the TD target is
  - State value

$$\theta \leftarrow \theta + \alpha(V^{\pi}(s_t) - V_\theta(s_t))\frac{\partial V_\theta(s_t)}{\partial \theta}$$

$$= \theta + \alpha(r_{t+1} + \gamma V_\theta(s_{t+1}) - V_\theta(s))\frac{\partial V_\theta(s_t)}{\partial \theta}$$

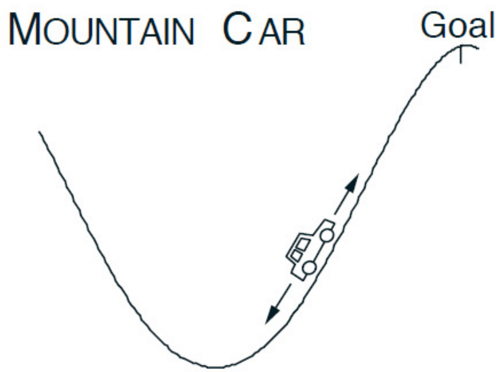
  - Action value

$$\theta \leftarrow \theta + \alpha(Q^{\pi}(s, a) - Q_\theta(s, a))\frac{\partial Q_\theta(s, a)}{\partial \theta}$$

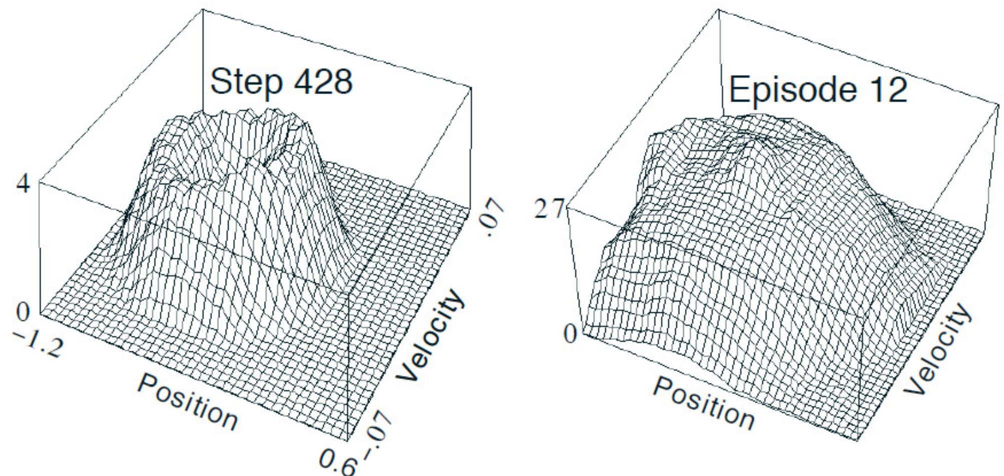
$$= \theta + \alpha(r_{t+1} + \gamma Q_\theta(s_{t+1}, a_{t+1}) - Q_\theta(s, a))\frac{\partial Q_\theta(s, a)}{\partial \theta}$$

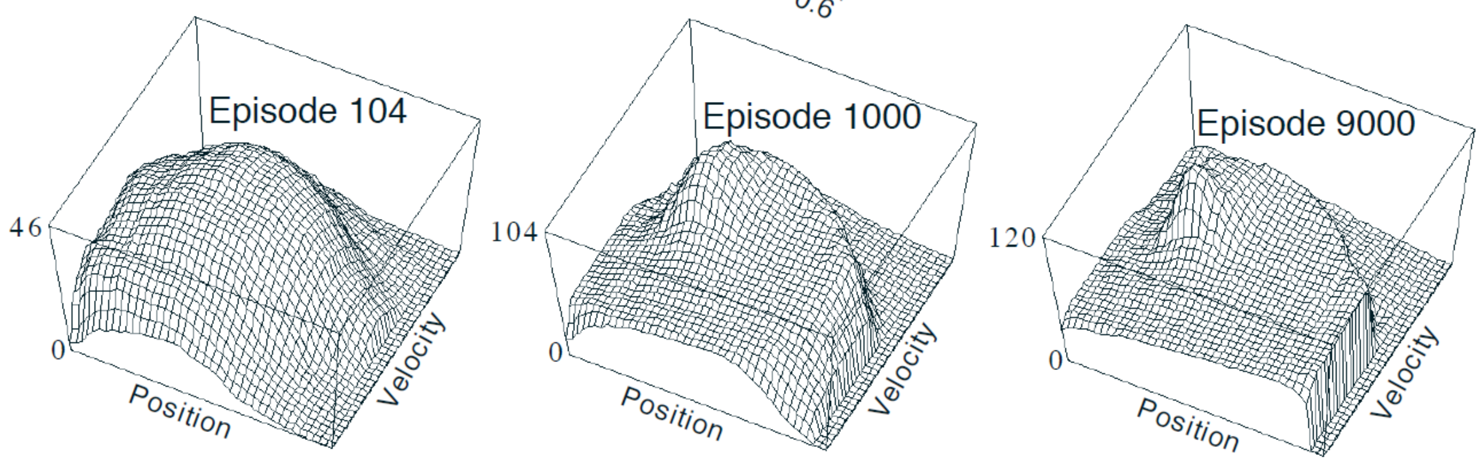
- Although ϑ is in the TD target, we don't calculate gradient from the target. Think about why.

Case Study: Mountain Car



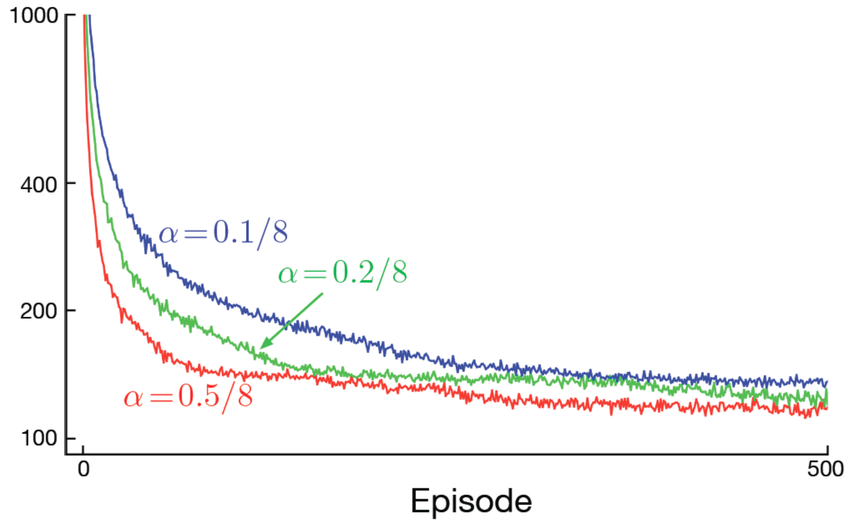
The gravity is stronger than the car's engine



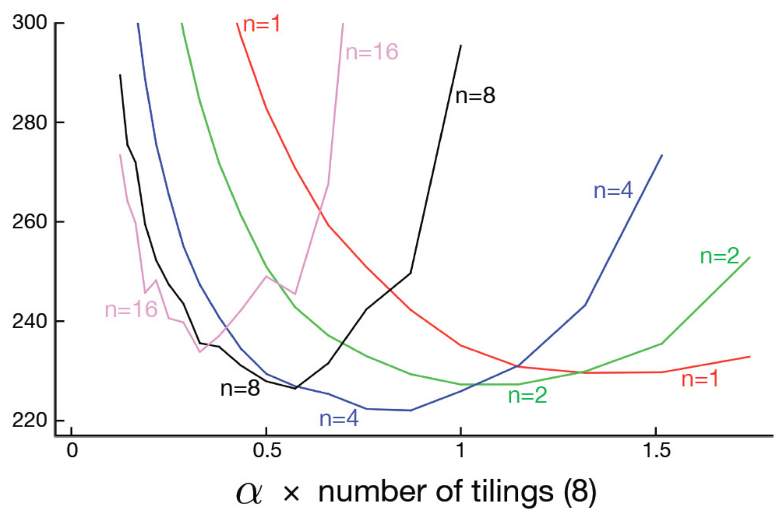


Ablation Study

Mountain Car
Steps per episode
log scale
averaged over 100 runs



Mountain Car
Steps per episode
averaged over
first 50 episodes
and 100 runs



Content

- Solutions for large MDPs
  - Discretize or bucketize states/actions
  - Build parametric value function approximation

- Policy gradient

- Deep reinforcement learning

Parametric Policy

- We can parametrize the policy

$$\pi_\theta(a|s)$$

which could be deterministic

$$a = \pi_\theta(s)$$

or stochastic

$$\pi_\theta(a|s) = P(a|s;\theta)$$

  - ϑ is the parameters of the policy
  - Generalize from seen states to unseen states
  - We focus on model-free reinforcement learning

Policy-based RL

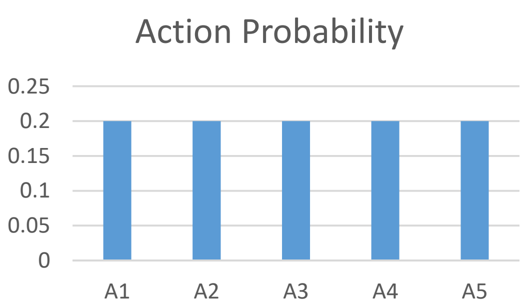
- Advantages
  - Better convergence properties
  - Effective in high-dimensional or continuous action spaces
  - Can learn stochastic polices

- Disadvantages
  - Typically converge to a local rather than global optimum
  - Evaluating a policy is typically inefficient and of high variance

Policy Gradient

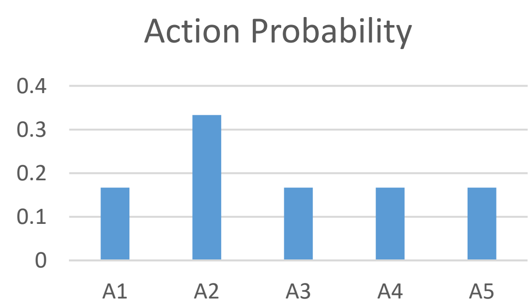
- For stochastic policy $\pi_\theta(a|s) = P(a|s;\theta)$
- Intuition
  - lower the probability of the action that leads to low value/reward
  - higher the probability of the action that leads to high value/reward
- A 5-action example

1. Initialize ϑ



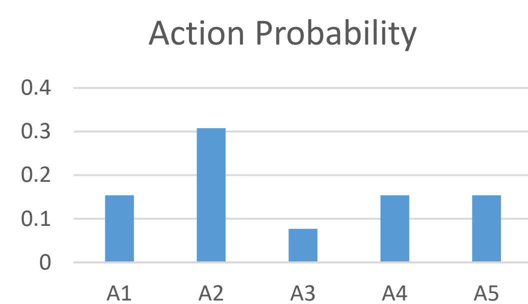
2. Take action A2
Observe positive reward

3. Update ϑ by policy gradient



4. Take action A3
Observe negative reward

5. Update ϑ by policy gradient



Policy Gradient in One-Step MDPs

- Consider a simple class of one-step MDPs
  - Starting in state $s \sim d(s)$
  - Terminating after one time-step with reward r_{sa}

- Policy expected value

$$J(\theta) = \mathbb{E}_{\pi_\theta}[r] = \sum_{s \in S} d(s) \sum_{a \in A} \pi_\theta(a|s) r_{sa}$$

$$\frac{\partial J(\theta)}{\partial \theta} = \sum_{s \in S} d(s) \sum_{a \in A} \frac{\partial \pi_\theta(a|s)}{\partial \theta} r_{sa}$$

Likelihood Ratio

- Likelihood ratios exploit the following identity

$$\frac{\partial \pi_\theta(a|s)}{\partial \theta} = \pi_\theta(a|s) \frac{1}{\pi_\theta(a|s)} \frac{\partial \pi_\theta(a|s)}{\partial \theta}$$
$$= \pi_\theta(a|s) \frac{\partial \log \pi_\theta(a|s)}{\partial \theta}$$

- Thus the policy's expected value

$$J(\theta) = \mathbb{E}_{\pi_\theta}[r] = \sum_{s \in S} d(s) \sum_{a \in A} \pi_\theta(a|s) \boxed{r_{sa}}$$

$$\frac{\partial J(\theta)}{\partial \theta} = \sum_{s \in S} d(s) \sum_{a \in A} \frac{\partial \pi_\theta(a|s)}{\partial \theta} r_{sa}$$

$$= \sum_{s \in S} d(s) \sum_{a \in A} \pi_\theta(a|s) \boxed{\frac{\partial \log \pi_\theta(a|s)}{\partial \theta} r_{sa}}$$

$$= \mathbb{E}_{\pi_\theta}\left[\frac{\partial \log \pi_\theta(a|s)}{\partial \theta} r_{sa}\right]$$

This can be approximated by sampling state *s* from *d*(*s*) and action *a* from π_ϑ

Policy Gradient Theorem

- The policy gradient theorem generalizes the likelihood ratio approach to multi-step MDPs
    - Replaces instantaneous reward r_{sa} with long-term value $Q^{\pi_\theta}(s,a)$
- Policy gradient theorem applies to
    - start state objective, average reward objective, and average value objective
- Theorem
    - For any differentiable policy $\pi_\theta(a|s)$, for any of policy objective function $J = J_1, J_{avR}, J_{avV}$, the policy gradient is

$$\frac{\partial J(\theta)}{\partial \theta} = \mathbb{E}_{\pi_\theta}\Big[\frac{\partial \log \pi_\theta(a|s)}{\partial \theta} Q^{\pi_\theta}(s,a)\Big]$$

Please refer to appendix of the slides for detailed proofs

Monte-Carlo Policy Gradient (REINFORCE)

- Update parameters by stochastic gradient ascent
- Using policy gradient theorem
- Using return v_t as an unbiased sample of $Q^{\pi_\theta}(s,a)$

$$\Delta\theta_t = \alpha \frac{\partial \log \pi_\theta(a_t|s_t)}{\partial\theta} v_t$$

- REINFORCE Algorithm

  Initialize ϑ arbitrarily

  for each episode $\{s_1, a_1, r_2, \ldots, s_{T-1}, a_{T-1}, r_T\} \sim \pi_\theta$ do

  for t=1 to T-1 do

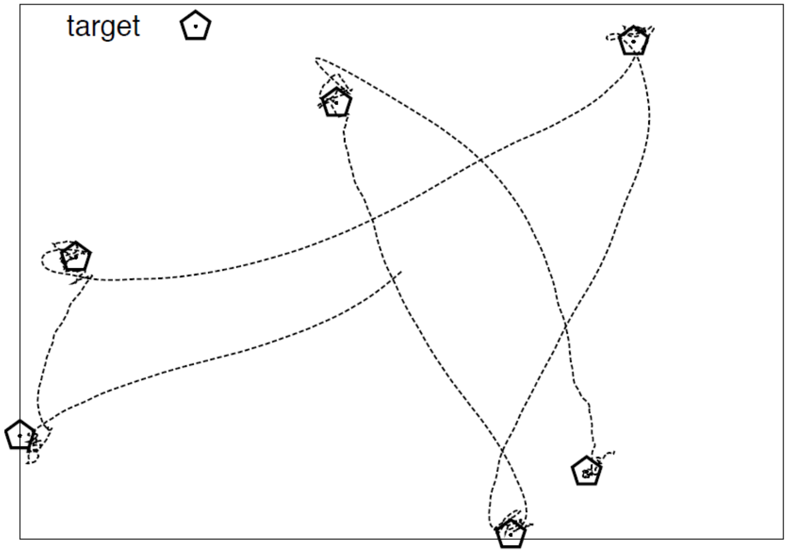
  $\theta \leftarrow \theta + \alpha \frac{\partial}{\partial\theta} \log \pi_\theta(a_t|s_t) v_t$

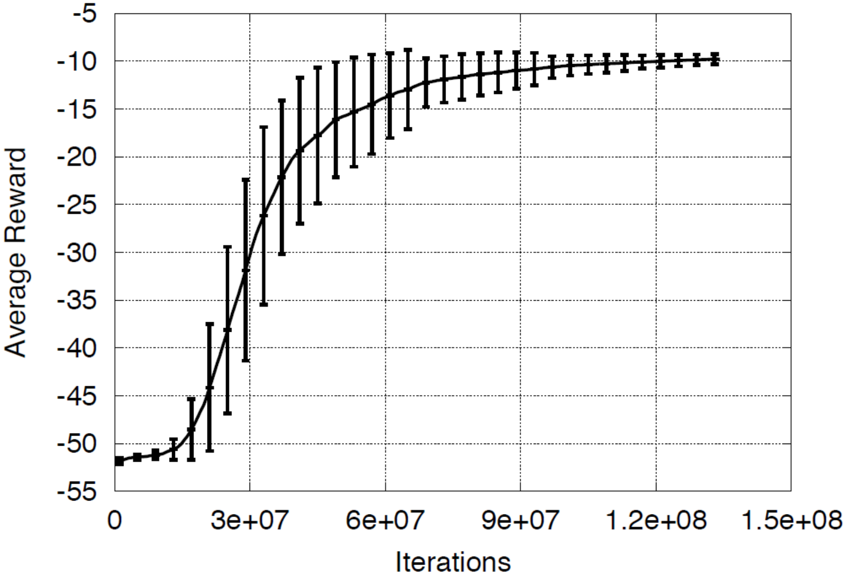
  end for

  end for

  return ϑ

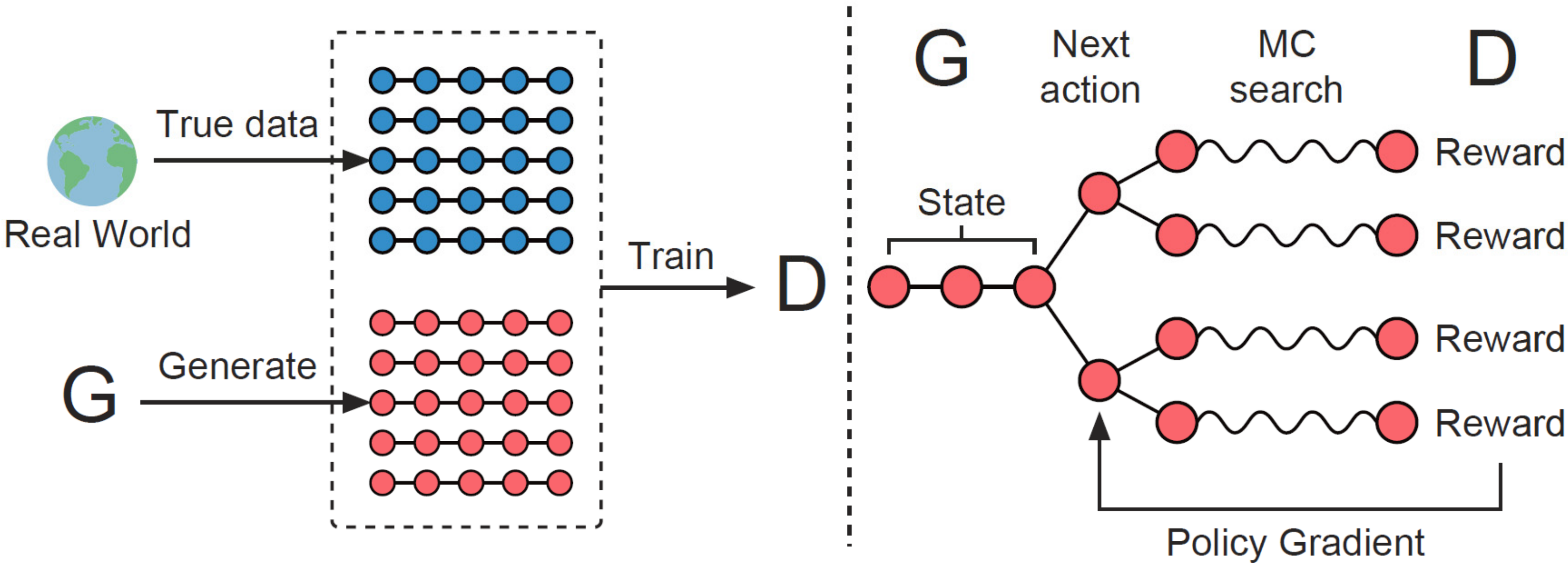
Puck World Example





- Continuous actions exert small force on puck
- Puck is rewarded for getting close to target
- Target location is reset every 30 seconds
- Policy is trained using variant of MC policy gradient

Sequence Generation Example



- Generator is a reinforcement learning policy $G_\theta(y_t|Y_{1:t-1})$ of generating a sequence
  - Decide the next word to generate given the previous ones
  - Discriminator provides the reward (i.e. the probability of being true data) for the whole sequence
  - G is trained by MC policy gradient (REINFORCE)

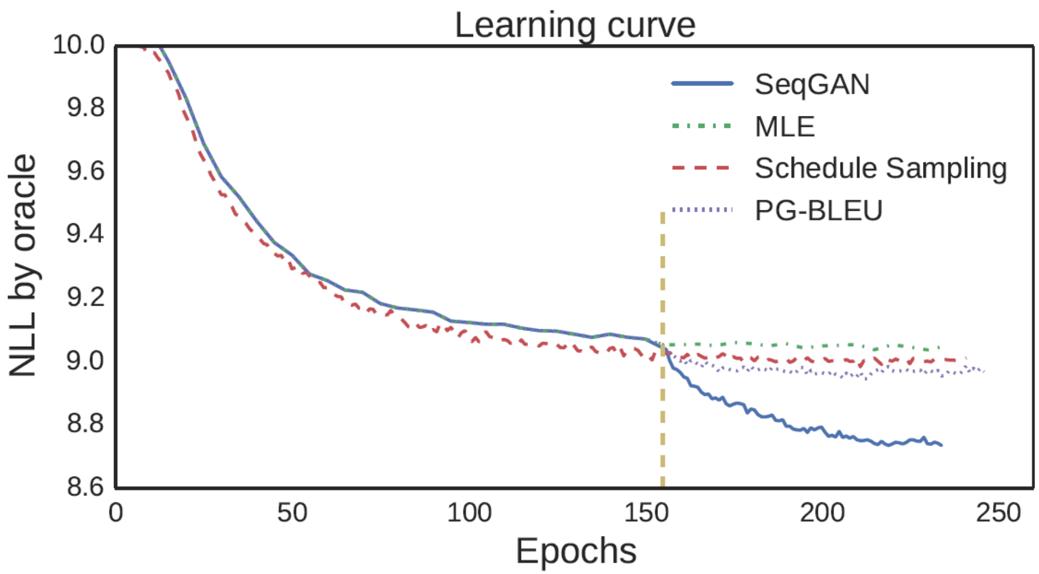
[Lantao Yu, Weinan Zhang, Jun Wang, Yong Yu. SeqGAN: Sequence Generative Adversarial Nets with Policy Gradient. AAAI 2017.]

Experiments on Synthetic Data

- Evaluation measure with Oracle

$$\text{NLL}_{\text{oracle}} = -\mathbb{E}_{Y_{1:T}\sim G_\theta}\Big[\sum_{t=1}^{T} \log G_{\text{oracle}}(y_t|Y_{1:t-1})\Big]$$

Algorithm	Random	MLE	SS	PG-BLEU	SeqGAN
NLL	10.310	9.038	8.985	8.946	**8.736**
p-value	$< 10^{-6}$	$< 10^{-6}$	$< 10^{-6}$	$< 10^{-6}$	



[Lantao Yu, Weinan Zhang, Jun Wang, Yong Yu. SeqGAN: Sequence Generative Adversarial Nets with Policy Gradient. AAAI 2017.]

Softmax Stochastic Policy

- Softmax policy is a very commonly used stochastic policy

$$\pi_\theta(a|s) = \frac{e^{f_\theta(s,a)}}{\sum_{a'} e^{f_\theta(s,a')}}$$

  - where $f_\vartheta(s,a)$ is the score function of a state-action pair parametrized by ϑ, which can be defined with domain knowledge

- The gradient of its log-likelihood

$$\frac{\partial \log \pi_\theta(a|s)}{\partial \theta} = \frac{\partial f_\theta(s,a)}{\partial \theta} - \frac{1}{\sum_{a'} e^{f_\theta(s,a')}} \sum_{a''} e^{f_\theta(s,a'')} \frac{\partial f_\theta(s,a'')}{\partial \theta}$$

$$= \frac{\partial f_\theta(s,a)}{\partial \theta} - \mathbb{E}_{a' \sim \pi_\theta(a'|s)}\left[\frac{\partial f_\theta(s,a')}{\partial \theta}\right]$$

Softmax Stochastic Policy

- Softmax policy is a very commonly used stochastic policy

$$\pi_\theta(a|s) = \frac{e^{f_\theta(s,a)}}{\sum_{a'} e^{f_\theta(s,a')}}$$

  - where $f_\vartheta(s,a)$ is the score function of a state-action pair parametrized by ϑ, which can be defined with domain knowledge

- For example, we define the linear score function

$$f_\theta(s,a) = \theta^\top x(s,a)$$

$$\begin{aligned}\frac{\partial \log \pi_\theta(a|s)}{\partial \theta} &= \frac{\partial f_\theta(s,a)}{\partial \theta} - \mathbb{E}_{a' \sim \pi_\theta(a'|s)}\Big[\frac{\partial f_\theta(s,a')}{\partial \theta}\Big] \\ &= x(s,a) - \mathbb{E}_{a' \sim \pi_\theta(a'|s)}[x(s,a')]\end{aligned}$$

Content

- Solutions for large MDPs
  - Discretize or bucketize states/actions
  - Build parametric value function approximation
- Policy gradient
- Deep reinforcement learning
  - By our invited speaker Xiaohu Zhu

Deep Reinforcement Learning

- Xiaohu Zhu

APPENDIX

Policy Gradient Theorem: Average Reward Setting

- Average reward objective

$$J(\pi) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\Big[r_1 + r_2 + \cdots + r_n | \pi\Big] = \sum_s d^\pi(s) \sum_a \pi(s,a) r(s,a)$$

$$Q^\pi(s,a) = \sum_{t=1}^{\infty} \mathbb{E}\Big[r_t - J(\pi) | s_0 = s, a_0 = a, \pi\Big]$$

$$\begin{aligned}
\frac{\partial V^\pi(s)}{\partial \theta} &\stackrel{\text{def}}{=} \frac{\partial}{\partial \theta} \sum_a \pi(s,a) Q^\pi(s,a), \quad \forall s \\
&= \sum_a \Big[\frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a) + \pi(s,a) \frac{\partial}{\partial \theta} Q^\pi(s,a)\Big] \\
&= \sum_a \Big[\frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a) + \pi(s,a) \frac{\partial}{\partial \theta}\Big(r(s,a) - J(\pi) + \sum_{s'} P^a_{ss'} V^\pi(s')\Big)\Big] \\
&= \sum_a \Big[\frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a) + \pi(s,a)\Big(-\frac{\partial J(\pi)}{\partial \theta} + \frac{\partial}{\partial \theta} \sum_{s'} P^a_{ss'} V^\pi(s')\Big)\Big]
\end{aligned}$$

$$\Rightarrow \frac{\partial J(\pi)}{\partial \theta} = \sum_a \Big[\frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a) + \pi(s,a) \sum_{s'} P^a_{ss'} \frac{\partial V^\pi(s')}{\partial \theta}\Big] - \frac{\partial V^\pi(s)}{\partial \theta}$$

Please refer to Chapter 13 of Rich Sutton's Reinforcement Learning: An Introduction (2nd Edition)

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Policy Gradient Theorem: Average Reward Setting

- Average reward objective

$$\frac{\partial J(\pi)}{\partial \theta} = \sum_a \Big[\frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a) + \pi(s,a) \sum_{s'} P^a_{ss'} \frac{\partial V^\pi(s')}{\partial \theta}\Big] - \frac{\partial V^\pi(s)}{\partial \theta}$$

$$\sum_s d^\pi(s) \frac{\partial J(\pi)}{\partial \theta} = \sum_s d^\pi(s) \sum_a \frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a) + \sum_s d^\pi(s) \sum_a \pi(s,a) \sum_{s'} P^a_{ss'} \frac{\partial V^\pi(s')}{\partial \theta} - \sum_s d^\pi(s) \frac{\partial V^\pi(s)}{\partial \theta}$$

$$\sum_s d^\pi(s) \sum_a \pi(s,a) \sum_{s'} P^a_{ss'} \frac{\partial V^\pi(s')}{\partial \theta} = \sum_s \sum_a \sum_{s'} d^\pi(s) \pi(s,a) P^a_{ss'} \frac{\partial V^\pi(s')}{\partial \theta}$$

$$= \sum_s \sum_{s'} d^\pi(s) \Big(\sum_a \pi(s,a) P^a_{ss'}\Big) \frac{\partial V^\pi(s')}{\partial \theta} = \sum_s \sum_{s'} d^\pi(s) P_{ss'} \frac{\partial V^\pi(s')}{\partial \theta}$$

$$= \sum_{s'} \Big(\sum_s d^\pi(s) P_{ss'}\Big) \frac{\partial V^\pi(s')}{\partial \theta} = \sum_{s'} d^\pi(s') \frac{\partial V^\pi(s')}{\partial \theta}$$

$$\Rightarrow \sum_s d^\pi(s) \frac{\partial J(\pi)}{\partial \theta} = \sum_s d^\pi(s) \sum_a \frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a) + \sum_{s'} d^\pi(s') \frac{\partial V^\pi(s')}{\partial \theta} - \sum_s d^\pi(s) \frac{\partial V^\pi(s)}{\partial \theta}$$

$$\Rightarrow \frac{\partial J(\pi)}{\partial \theta} = \sum_s d^\pi(s) \sum_a \frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a)$$

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APPENDIX

Policy gradient theorem: Start Value Setting

- Start state value objective

$$J(\pi) = \mathbb{E}\Big[\sum_{t=1}^{\infty} \gamma^{t-1} r_t \Big| s_0, \pi\Big]$$

$$Q^\pi(s,a) = \mathbb{E}\Big[\sum_{k=1}^{\infty} \gamma^{k-1} r_{t+k} \Big| s_t = s, a_t = a, \pi\Big]$$

$$\begin{aligned}
\frac{\partial V^\pi(s)}{\partial \theta} &\stackrel{\text{def}}{=} \frac{\partial}{\partial \theta} \sum_a \pi(s,a) Q^\pi(s,a), \quad \forall s \\
&= \sum_a \Big[\frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a) + \pi(s,a) \frac{\partial}{\partial \theta} Q^\pi(s,a)\Big] \\
&= \sum_a \Big[\frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a) + \pi(s,a) \frac{\partial}{\partial \theta} \Big(r(s,a) + \sum_{s'} \gamma P^a_{ss'} V^\pi(s')\Big)\Big] \\
&= \sum_a \frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a) + \sum_a \pi(s,a) \gamma \sum_{s'} P^a_{ss'} \frac{\partial V^\pi(s')}{\partial \theta}
\end{aligned}$$

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APPENDIX

Policy gradient theorem: Start Value Setting

- Start state value objective

$$\frac{\partial V^\pi(s)}{\partial\theta} = \sum_a \frac{\partial\pi(s,a)}{\partial\theta}Q^\pi(s,a) + \sum_a \pi(s,a)\gamma\sum_{s_1} P^a_{ss_1}\frac{\partial V^\pi(s_1)}{\partial\theta}$$

$$\sum_a \frac{\partial\pi(s,a)}{\partial\theta}Q^\pi(s,a) = \gamma^0 Pr(s\rightarrow s,0,\pi)\sum_a \frac{\partial\pi(s,a)}{\partial\theta}Q^\pi(s,a)$$

$$\begin{aligned}\sum_a \pi(s,a)\gamma\sum_{s_1} P^a_{ss_1}\frac{\partial V^\pi(s_1)}{\partial\theta} &= \sum_{s_1}\sum_a \pi(s,a)\gamma P^a_{ss_1}\frac{\partial V^\pi(s_1)}{\partial\theta}\\ &= \sum_{s_1}\gamma P_{ss_1}\frac{\partial V^\pi(s_1)}{\partial\theta} = \gamma^1\sum_{s_1} Pr(s\rightarrow s_1,1,\pi)\frac{\partial V^\pi(s_1)}{\partial\theta}\end{aligned}$$

$$\frac{\partial V^\pi(s_1)}{\partial\theta} = \sum_a \frac{\partial\pi(s,a)}{\partial\theta}Q^\pi(s,a) + \gamma^1\sum_{s_2} Pr(s_1\rightarrow s_2,1,\pi)\frac{\partial V^\pi(s_2)}{\partial\theta}$$

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APPENDIX

Policy gradient theorem: Start Value Setting

- Start state value objective

$$\frac{\partial V^\pi(s)}{\partial \theta} = \gamma^0 Pr(s \to s, 0, \pi) \sum_a \frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a) + \gamma^1 \sum_{s_1} Pr(s \to s_1, 1, \pi) \sum_a \frac{\partial \pi(s_1,a)}{\partial \theta} Q^\pi(s_1,a)$$

$$+ \gamma^2 \sum_{s_1} Pr(s \to s_1, 1, \pi) \sum_{s_2} Pr(s_1 \to s_2, 1, \pi) \frac{\partial V^\pi(s_2)}{\partial \theta}$$

$$= \gamma^0 Pr(s \to s, 0, \pi) \sum_a \frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a) + \gamma^1 \sum_{s_1} Pr(s \to s_1, 1, \pi) \sum_a \frac{\partial \pi(s_1,a)}{\partial \theta} Q^\pi(s_1,a)$$

$$+ \gamma^2 \sum_{s_2} Pr(s \to s_2, 2, \pi) \frac{\partial V^\pi(s_2)}{\partial \theta}$$

$$= \sum_{k=0}^{\infty} \sum_x \gamma^k Pr(s \to x, k, \pi) \sum_a \frac{\partial \pi(x,a)}{\partial \theta} Q^\pi(x,a) = \sum_x \sum_{k=0}^{\infty} \gamma^k Pr(s \to x, k, \pi) \sum_a \frac{\partial \pi(x,a)}{\partial \theta} Q^\pi(x,a)$$

$$\Rightarrow \frac{\partial J(\pi)}{\partial \theta} = \frac{\partial V^\pi(s_0)}{\partial \theta} = \sum_s \sum_{k=0}^{\infty} \gamma^k Pr(s_0 \to s, k, \pi) \sum_a \frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a) = \sum_s d^\pi(s) \sum_a \frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a)$$

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