2019 CS420, Machine Learning, Lecture 12

Approximation Methods in Reinforcement Learning

Weinan Zhang

Shanghai Jiao Tong University

http://wnzhang.net

http://wnzhang.net/teaching/cs420/index.html

Reinforcement Learning Materials

Our course on RL is mainly based on the materials from these masters.

Prof. Richard Sutton

- University of Alberta, Canada
- http://incompleteideas.net/sutton/index.html
- Reinforcement Learning: An Introduction (2nd edition)
- http://incompleteideas.net/sutton/book/the-book-2nd.html

Dr. David Silver

- Google DeepMind and UCL, UK
- http://www0.cs.ucl.ac.uk/staff/d.silver/web/Home.html
- UCL Reinforcement Learning Course
- http://www0.cs.ucl.ac.uk/staff/d.silver/web/Teaching.html

Prof. Andrew Ng

- Stanford University, US
- http://www.andrewng.org/
- Machine Learning (CS229) Lecture Notes 12: RL
- http://cs229.stanford.edu/materials.html

Last Lecture

- Model-based dynamic programming
  - Value iteration $V(s) = R(s) + \max_{a \in A} \gamma \sum_{s' \in S} P_{sa}(s')V(s')$
  - Policy iteration $\pi(s) = \arg\max_{a \in A} \sum_{s' \in S} P_{sa}(s')V(s')$
- Model-free reinforcement learning
  - On-policy MC $V(s_t) \leftarrow V(s_t) + \alpha(G_t - V(s_t))$
  - On-policy TD $V(s_t) \leftarrow V(s_t) + \alpha(r_{t+1} + \gamma V(s_{t+1}) - V(s_t))$
  - On-policy TD SARSA

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_{t+1} + \gamma Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t))$$

  - Off-policy TD Q-learning

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_{t+1} + \gamma \max_{a'} Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t))$$

Key Problem to Solve in This Lecture

- In all previous models, we have created a lookup table to maintain a variable $V(s)$ for each state or $Q(s,a)$ for each state-action

- What if we have a large MDP, i.e.
    - the state or state-action space is too large
    - or the state or action space is continuous

  to maintain $V(s)$ for each state or $Q(s,a)$ for each state-action?
    - For example
        - Game of Go (10^{170} states)
        - Helicopter, autonomous car (continuous state space)

Content

- Solutions for large MDPs
  - Discretize or bucketize states/actions
  - Build parametric value function approximation

- Policy gradient

- Deep reinforcement learning and multi-agent RL

Content

- Solutions for large MDPs
  - Discretize or bucketize states/actions
  - Build parametric value function approximation

- Policy gradient

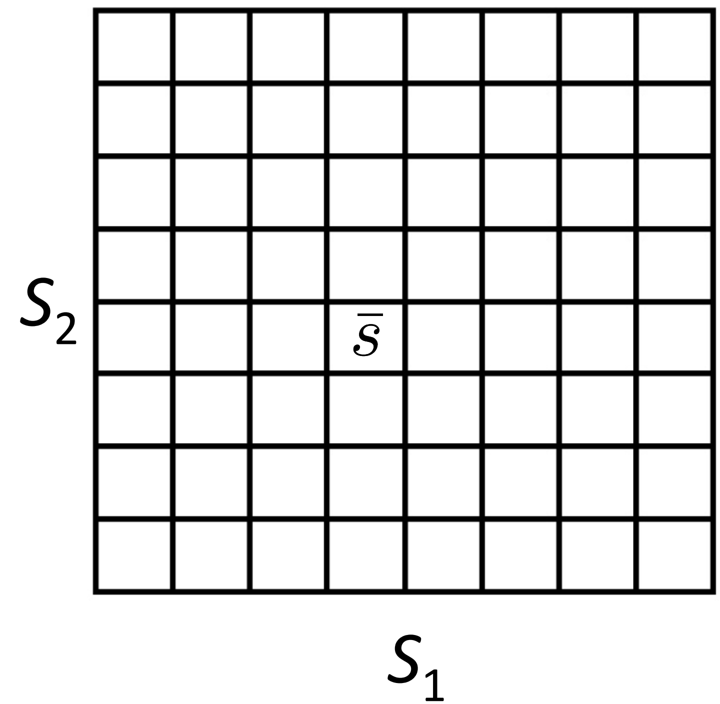
- Deep reinforcement learning and multi-agent RL

Discretization Continuous MDP

- For a continuous-state MDP, we can discretize the state space
  - For example, if we have 2D states (s_1, s_2), we can use a grid to discretize the state space
  - The discrete state $\bar{s}$
  - The discretized MDP:

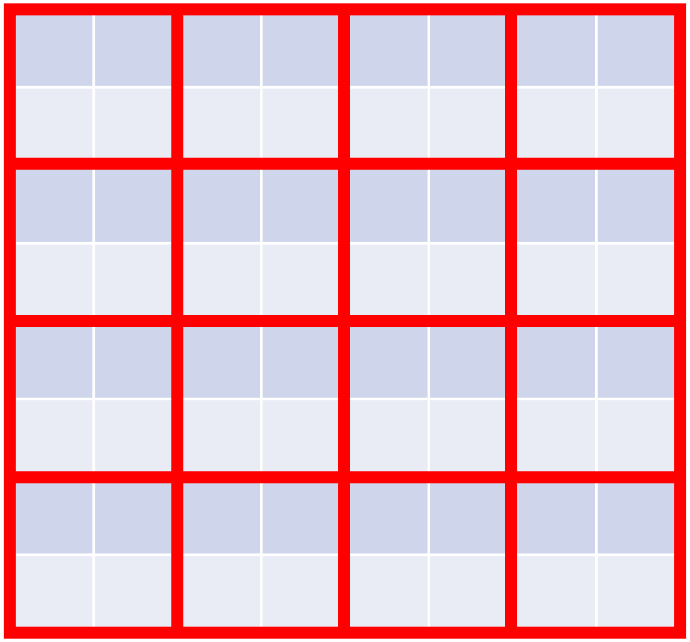
$$(\bar{S}, A, \{P_{\bar{s}a}\}, \gamma, R)$$

  - Then solve this MDP with any previous solutions



Bucketize Large Discrete MDP

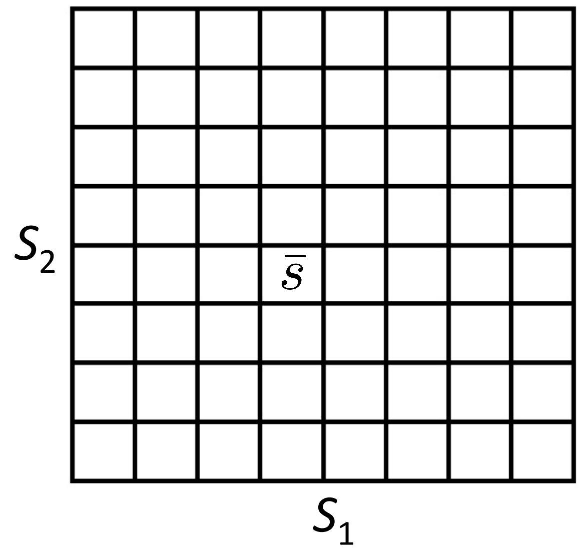
- For a large discrete-state MDP, we can bucketize the states to down sample the states
  - To use domain knowledge to merge similar discrete states
    - For example, clustering using state features extracted from domain knowledge

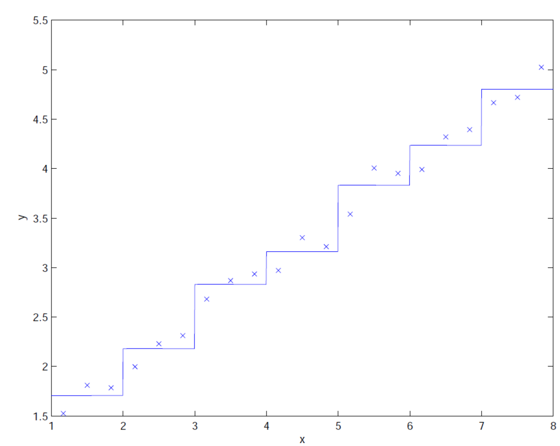


Discretization/Bucketization

- Pros
  - Straightforward and off-the-shelf
  - Efficient
  - Can work well for many problems
- Cons
  - A fairly naïve representation for *V*
  - Assumes a constant value over each discretized cell
  - Curse of dimensionality

$$S = \mathbb{R}^n \Rightarrow \bar{S} = \{1, \ldots, k\}^n$$





Parametric Value Function Approximation

- Create parametric (thus learnable) functions to approximate the value function

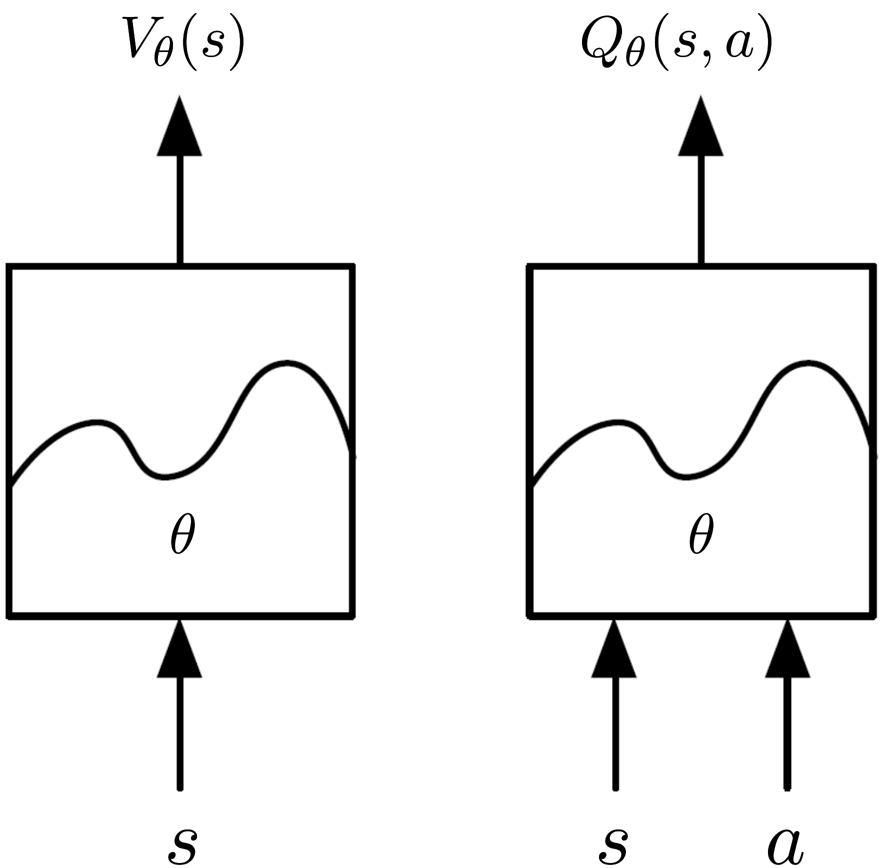
$$V_\theta(s) \simeq V^\pi(s)$$

$$Q_\theta(s,a) \simeq Q^\pi(s,a)$$

  - ϑ is the parameters of the approximation function, which can be updated by reinforcement learning

  - Generalize from seen states to unseen states

Main Types of Value Function Approx.



Many function approximations

- (Generalized) linear model
- Neural network
- Decision tree
- Nearest neighbor
- Fourier / wavelet bases

Differentiable functions

- (Generalized) linear model
- Neural network

We assume the model is suitable to be trained for non-stationary, non-iid data

Value Function Approx. by SGD

- Goal: find parameter vector ϑ minimizing mean-squared error between approximate value function $V_\vartheta(s)$ and true value $V^\pi(s)$

$$J(\theta) = \mathbb{E}_\pi \left[\frac{1}{2} (V^\pi(s) - V_\theta(s))^2 \right]$$

- Gradient to minimize the error

$$-\frac{\partial J(\theta)}{\partial \theta} = \mathbb{E}_\pi \left[(V^\pi(s) - V_\theta(s)) \frac{\partial V_\theta(s)}{\partial \theta} \right]$$

- Stochastic gradient descent on one sample

$$\begin{aligned} \theta &\leftarrow \theta - \alpha \frac{\partial J(\theta)}{\partial \theta} \\ &= \theta + \alpha (V^\pi(s) - V_\theta(s)) \frac{\partial V_\theta(s)}{\partial \theta} \end{aligned}$$

Featurize the State

- Represent state by a feature vector

$$x(s) = \begin{bmatrix} x_1(s) \\ \vdots \\ x_k(s) \end{bmatrix}$$

- For example of a helicopter
  - 3D location
  - 3D speed (differentiation of location)
  - 3D acceleration (differentiation of speed)

Linear Value Function Approximation

- Represent value function by a linear combination of features

$$V_\theta(s) = \theta^\top x(s)$$

- Objective function is quadratic in parameters $\boldsymbol{\vartheta}$

$$J(\theta) = \mathbb{E}_\pi \left[\frac{1}{2} (V^\pi(s) - \theta^\top x(s))^2 \right]$$

- Thus stochastic gradient descent converges on global optimum

$$\begin{aligned} \theta &\leftarrow \theta - \alpha \frac{\partial J(\theta)}{\partial \theta} \\ &= \theta + \underbrace{\alpha}_{\text{Step size}} (\underbrace{V^\pi(s) - V_\theta(s)}_{\text{Prediction error}}) \underbrace{x(s)}_{\text{Feature value}} \end{aligned}$$

Monte-Carlo with Value Function Approx.

$$\theta \leftarrow \theta + \alpha(V^{\pi}(s) - V_{\theta}(s))x(s)$$

- Now we specify the target value function $V^{\pi}(s)$
- We can apply supervised learning to "training data"

$$\langle s_1, G_1 \rangle, \langle s_2, G_2 \rangle, \ldots, \langle s_T, G_T \rangle$$

- For each data instance $<s_t, G_t>$

$$\theta \leftarrow \theta + \alpha(G_t - V_{\theta}(s))x(s_t)$$

- MC evaluation at least converges to a local optimum
  - In linear case it converges to a global optimum

TD Learning with Value Function Approx.

$$\theta \leftarrow \theta + \alpha(V^{\pi}(s) - V_{\theta}(s))x(s)$$

- TD target $r_{t+1} + \gamma V_{\theta}(s_{t+1})$ is a biased sample of true target value $V^{\pi}(s_t)$
- Supervised learning from "training data"

$$\langle s_1, r_2 + \gamma V_{\theta}(s_2) \rangle, \langle s_2, r_3 + \gamma V_{\theta}(s_3) \rangle, \ldots, \langle s_T, r_T \rangle$$

- For each data instance $\langle s_t, r_{t+1} + \gamma V_{\theta}(s_{t+1}) \rangle$

$$\theta \leftarrow \theta + \alpha(r_{t+1} + \gamma V_{\theta}(s_{t+1}) - V_{\theta}(s))x(s_t)$$

- Linear TD converges (close) to global optimum

Action-Value Function Approximation

- Approximate the action-value function

$$Q_\theta(s,a) \simeq Q^\pi(s,a)$$

- Minimize mean squared error

$$J(\theta) = \mathbb{E}_\pi\Big[\frac{1}{2}(Q^\pi(s,a) - Q_\theta(s,a))^2\Big]$$

- Stochastic gradient descent on one sample

$$\begin{aligned}\theta &\leftarrow \theta - \alpha\frac{\partial J(\theta)}{\partial\theta} \\ &= \theta + \alpha(Q^\pi(s,a) - Q_\theta(s,a))\frac{\partial Q_\theta(s,a)}{\partial\theta}\end{aligned}$$

Linear Action-Value Function Approx.

- Represent state-action pair by a feature vector

$$x(s,a) = \begin{bmatrix} x_1(s,a) \\ \vdots \\ x_k(s,a) \end{bmatrix}$$

- Parametric Q function, e.g., the linear case

$$Q_\theta(s,a) = \theta^\top x(s,a)$$

- Stochastic gradient descent update

$$\begin{aligned} \theta &\leftarrow \theta - \alpha \frac{\partial J(\theta)}{\partial \theta} \\ &= \theta + \alpha(Q^\pi(s,a) - \theta^\top x(s,a))x(s,a) \end{aligned}$$

TD Learning with Value Function Approx.

$$\theta \leftarrow \theta + \alpha(Q^\pi(s,a) - Q_\theta(s,a))\frac{\partial Q_\theta(s,a)}{\partial \theta}$$

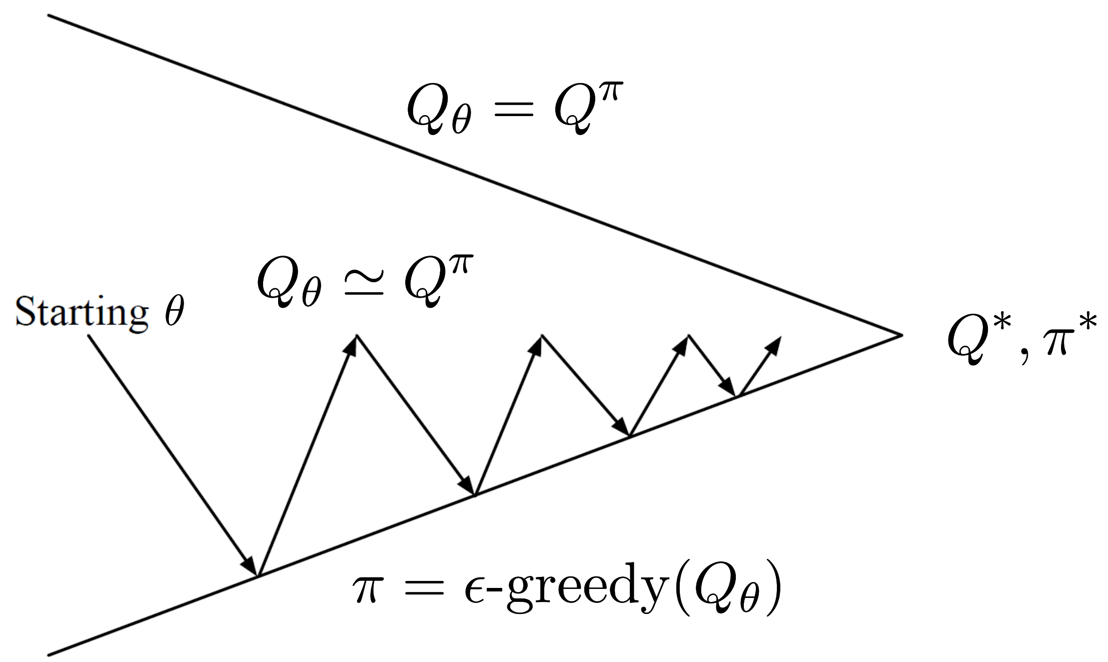
- For MC, the target is the return G_t

$$\theta \leftarrow \theta + \alpha(G_t - Q_\theta(s,a))\frac{\partial Q_\theta(s,a)}{\partial \theta}$$

- For TD, the target is $r_{t+1} + \gamma Q_\theta(s_{t+1}, a_{t+1})$

$$\theta \leftarrow \theta + \alpha(r_{t+1} + \gamma Q_\theta(s_{t+1}, a_{t+1}) - Q_\theta(s,a))\frac{\partial Q_\theta(s,a)}{\partial \theta}$$

Control with Value Function Approx.



- Policy evaluation: approximately policy evaluation $Q_\theta \simeq Q^\pi$
- Policy improvement: *ε*-greedy policy improvement

NOTE of TD Update

- For TD(0), the TD target is
  - State value

$$\theta \leftarrow \theta + \alpha(V^\pi(s_t) - V_\theta(s_t))\frac{\partial V_\theta(s_t)}{\partial\theta}$$

$$= \theta + \alpha(r_{t+1} + \gamma V_\theta(s_{t+1}) - V_\theta(s))\frac{\partial V_\theta(s_t)}{\partial\theta}$$

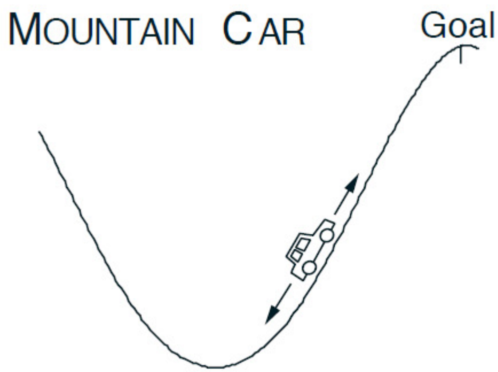
  - Action value

$$\theta \leftarrow \theta + \alpha(Q^\pi(s,a) - Q_\theta(s,a))\frac{\partial Q_\theta(s,a)}{\partial\theta}$$

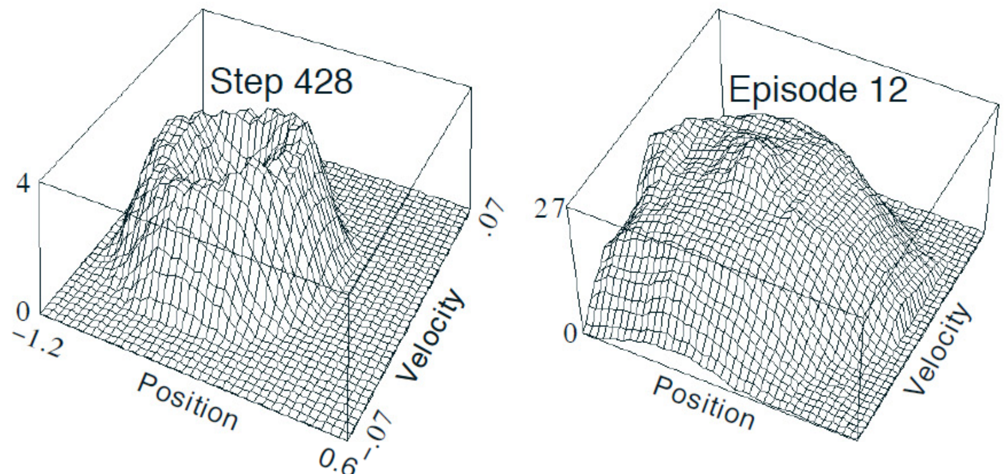
$$= \theta + \alpha(r_{t+1} + \gamma Q_\theta(s_{t+1}, a_{t+1}) - Q_\theta(s,a))\frac{\partial Q_\theta(s,a)}{\partial\theta}$$

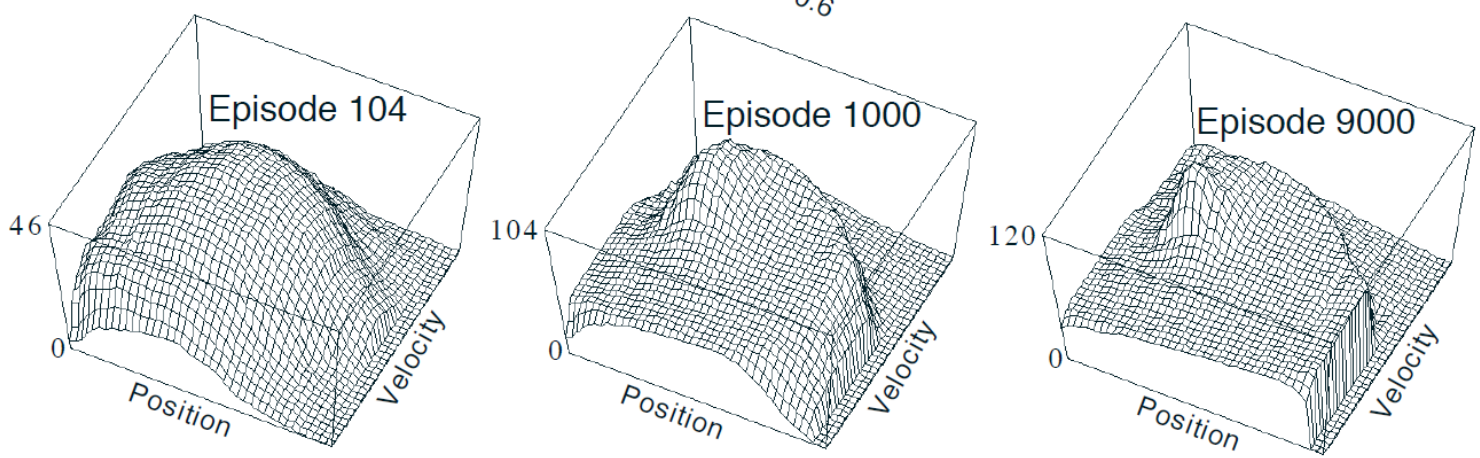
- Although ϑ is in the TD target, we don't calculate gradient from the target. Think about why.

Case Study: Mountain Car



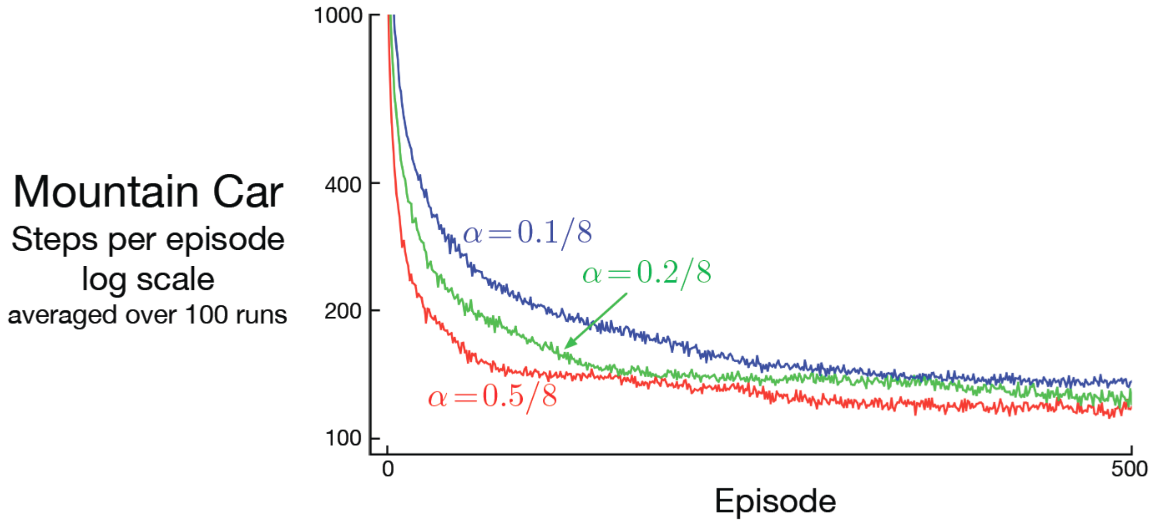
The gravity is stronger than the car's engine

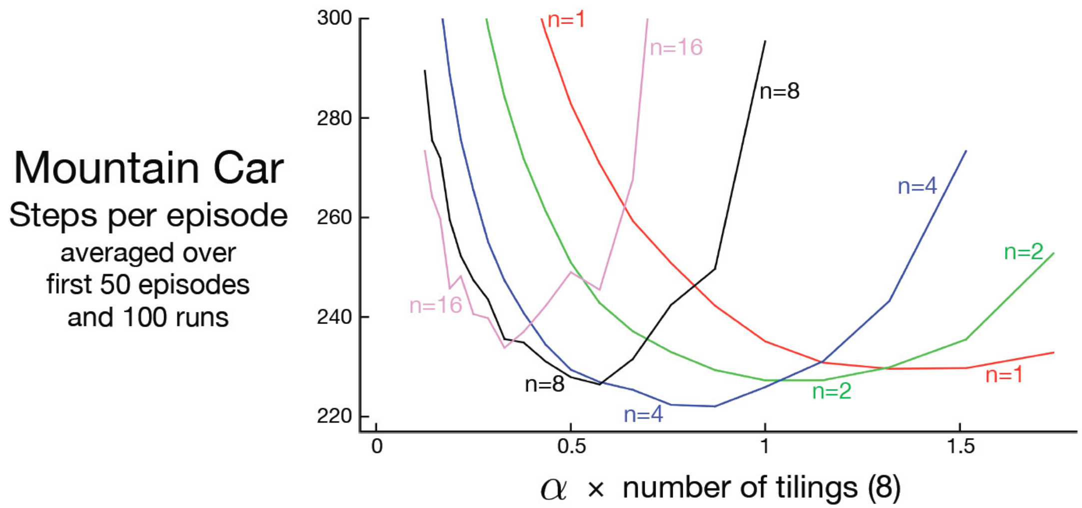




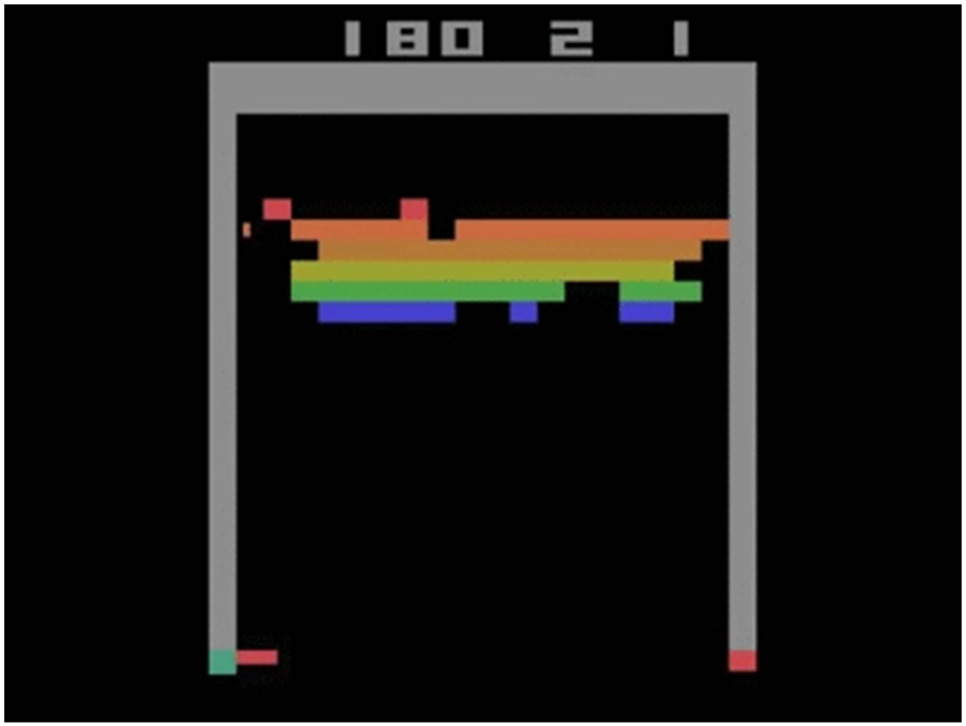
Cost-to-go function

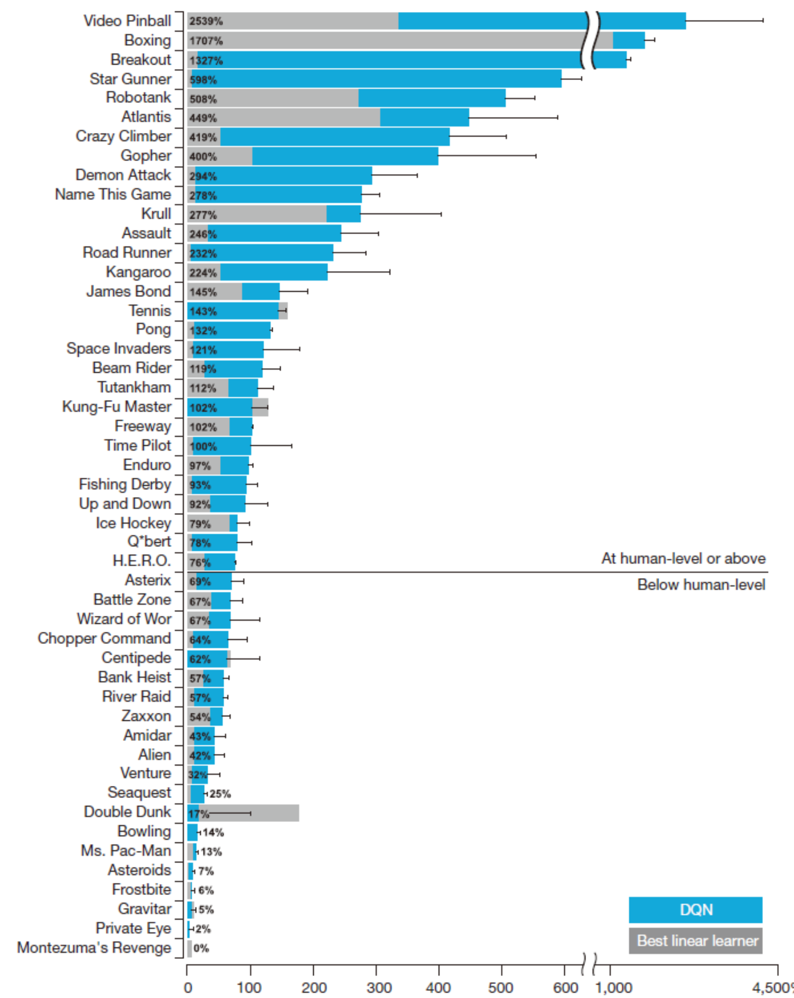
Case Study: Mountain Car





Deep Q-Network (DQN)

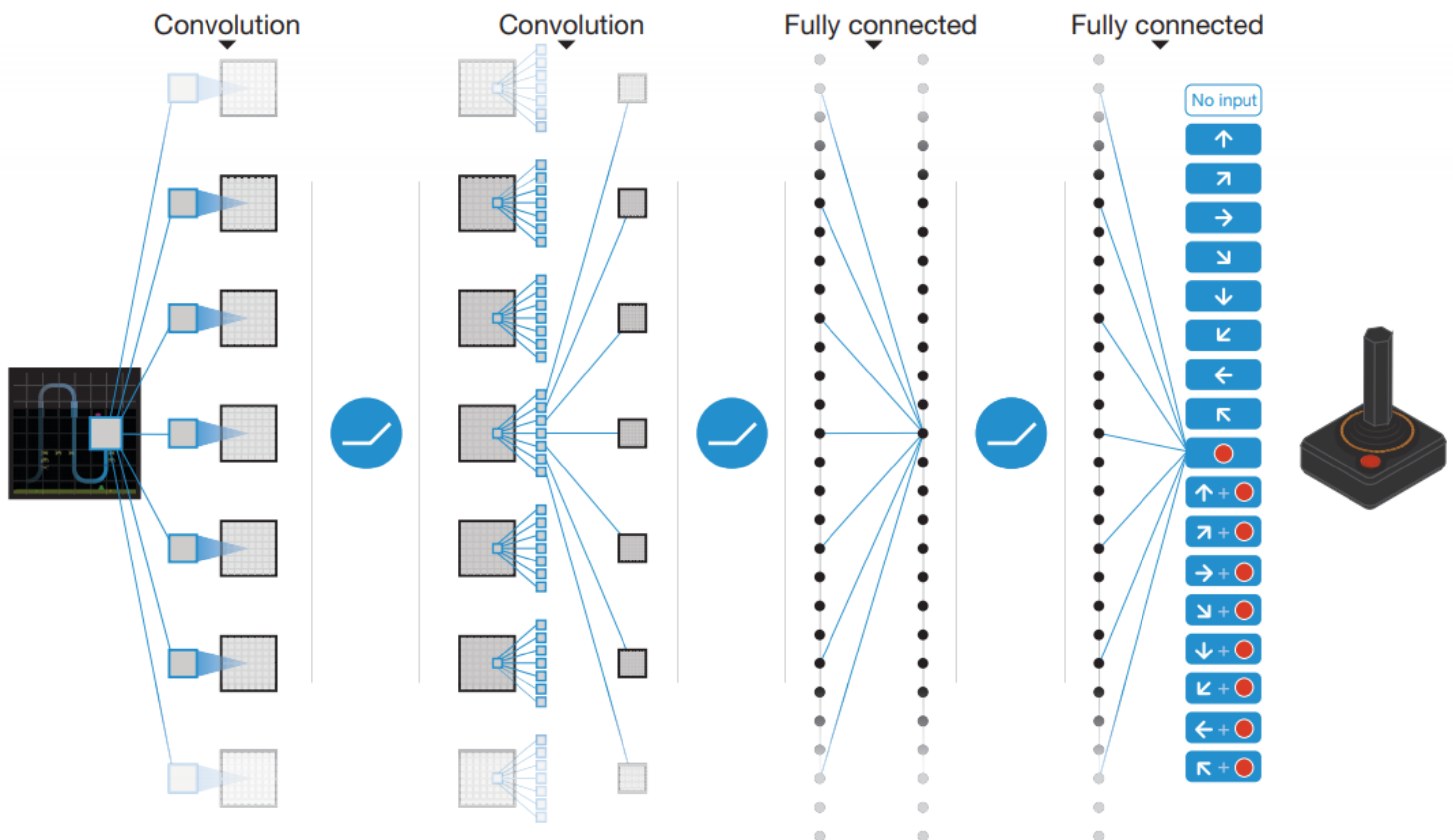




Volodymyr Mnih, Koray Kavukcuoglu, David Silver et al. Playing Atari with Deep Reinforcement Learning. NIPS 2013 workshop.
Volodymyr Mnih, Koray Kavukcuoglu, David Silver et al. Human-level control through deep reinforcement learning. Nature 2015.

Deep Q-Network (DQN)

- Implement Q function with deep neural network



Volodymyr Mnih, Koray Kavukcuoglu, David Silver et al. Playing Atari with Deep Reinforcement Learning. NIPS 2013 workshop.
Volodymyr Mnih, Koray Kavukcuoglu, David Silver et al. Human-level control through deep reinforcement learning. Nature 2015.

Deep Q-Network (DQN)

- The loss function of Q-learning update at iteration *i*

$$L_i(\theta_i) = \mathbb{E}_{(s,a,r,s')\sim U(D)}\Big[\big(\underbrace{r + \gamma \max_{a'} Q(s',a';\theta_i^-)}_{\text{target Q value}} - \underbrace{Q(s,a;\theta_i)}_{\text{estimated Q value}}\big)^2\Big]$$

  - ϑ_i are the network parameters to be updated at iteration *i*
    - Updated with standard back-propagation algorithms
  - ϑ_i^- are the target network parameters
    - Only updated with ϑ_i for every *C* steps
  - (*s*,*a*,*r*,*s'*)~*U*(*D*): the samples are uniformly drawn from the experience pool *D*
    - Thus to avoid the overfitting to the recent experiences

Volodymyr Mnih, Koray Kavukcuoglu, David Silver et al. Playing Atari with Deep Reinforcement Learning. NIPS 2013 workshop.
Volodymyr Mnih, Koray Kavukcuoglu, David Silver et al. Human-level control through deep reinforcement learning. Nature 2015.

Content

- Solutions for large MDPs
  - Discretize or bucketize states/actions
  - Build parametric value function approximation
- Policy gradient
- Deep reinforcement learning and multi-agent RL

Parametric Policy

- We can parametrize the policy

$$\pi_\theta(a|s)$$

which could be deterministic

$$a = \pi_\theta(s)$$

or stochastic

$$\pi_\theta(a|s) = P(a|s;\theta)$$

  - ϑ is the parameters of the policy
  - Generalize from seen states to unseen states
  - We focus on model-free reinforcement learning

Policy-based RL

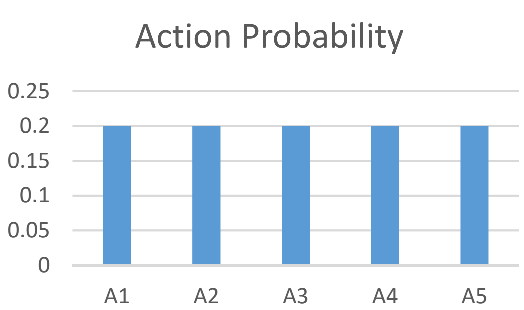
- Advantages
  - Better convergence properties
  - Effective in high-dimensional or continuous action spaces
    - No.1 reason: for value function, you have to take a max operation
  - Can learn stochastic polices

- Disadvantages
  - Typically converge to a local rather than global optimum
  - Evaluating a policy is typically inefficient and of high variance

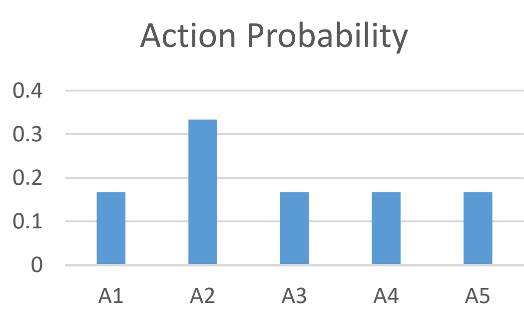
Policy Gradient

- For stochastic policy $\pi_\theta(a|s) = P(a|s;\theta)$
- Intuition
  - lower the probability of the action that leads to low value/reward
  - higher the probability of the action that leads to high value/reward
- A 5-action example

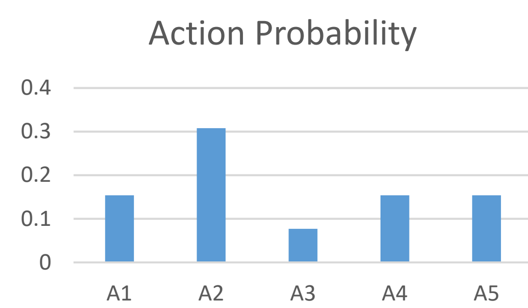
1. Initialize ϑ



3. Update ϑ by policy gradient



5. Update ϑ by policy gradient



2. Take action A2
Observe positive reward

4. Take action A3
Observe negative reward

Policy Gradient in One-Step MDPs

- Consider a simple class of one-step MDPs
  - Starting in state $s \sim d(s)$
  - Terminating after one time-step with reward r_{sa}

- Policy expected value

$$J(\theta) = \mathbb{E}_{\pi_\theta}[r] = \sum_{s \in S} d(s) \sum_{a \in A} \pi_\theta(a|s) r_{sa}$$

$$\frac{\partial J(\theta)}{\partial \theta} = \sum_{s \in S} d(s) \sum_{a \in A} \frac{\partial \pi_\theta(a|s)}{\partial \theta} r_{sa}$$

Likelihood Ratio

- Likelihood ratios exploit the following identity

$$\frac{\partial \pi_\theta(a|s)}{\partial \theta} = \pi_\theta(a|s)\frac{1}{\pi_\theta(a|s)}\frac{\partial \pi_\theta(a|s)}{\partial \theta}$$
$$= \pi_\theta(a|s)\frac{\partial \log \pi_\theta(a|s)}{\partial \theta}$$

- Thus the policy's expected value

$$J(\theta) = \mathbb{E}_{\pi_\theta}[r] = \sum_{s\in S} d(s) \sum_{a\in A} \pi_\theta(a|s)\boxed{r_{sa}}$$
$$\frac{\partial J(\theta)}{\partial \theta} = \sum_{s\in S} d(s) \sum_{a\in A} \frac{\partial \pi_\theta(a|s)}{\partial \theta} r_{sa}$$
$$= \sum_{s\in S} d(s) \sum_{a\in A} \pi_\theta(a|s)\boxed{\frac{\partial \log \pi_\theta(a|s)}{\partial \theta} r_{sa}}$$
$$= \mathbb{E}_{\pi_\theta}\left[\frac{\partial \log \pi_\theta(a|s)}{\partial \theta} r_{sa}\right]$$

This can be approximated by sampling state *s* from *d*(*s*) and action *a* from π_ϑ

Policy Gradient Theorem

- The policy gradient theorem generalizes the likelihood ratio approach to multi-step MDPs
  - Replaces instantaneous reward r_{sa} with long-term value $Q^{\pi_\theta}(s,a)$
- Policy gradient theorem applies to
  - start state objective J_1, average reward objective J_{avR}, and average value objective J_{avV}
- Theorem
  - For any differentiable policy $\pi_\theta(a|s)$, for any of policy objective function $J = J_1, J_{avR}, J_{avV}$, the policy gradient is

$$\frac{\partial J(\theta)}{\partial \theta} = \mathbb{E}_{\pi_\theta}\Big[\frac{\partial \log \pi_\theta(a|s)}{\partial \theta} Q^{\pi_\theta}(s,a)\Big]$$

Please refer to appendix of the slides for detailed proofs

Monte-Carlo Policy Gradient (REINFORCE)

- Update parameters by stochastic gradient ascent
- Using policy gradient theorem
- Using return G_t as an unbiased sample of $Q^{\pi_\theta}(s,a)$

$$\Delta\theta_t = \alpha \frac{\partial \log \pi_\theta(a_t|s_t)}{\partial \theta} G_t$$

- REINFORCE Algorithm

  Initialize ϑ arbitrarily

  for each episode $\{s_1, a_1, r_2, \ldots, s_{T-1}, a_{T-1}, r_T\} \sim \pi_\theta$ do

  for t=1 to T-1 do

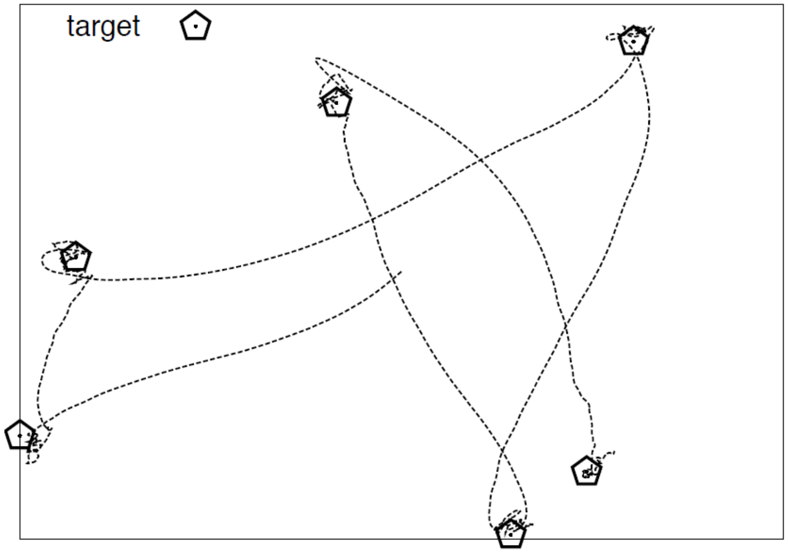
  $$\theta \leftarrow \theta + \alpha \frac{\partial}{\partial \theta} \log \pi_\theta(a_t|s_t) G_t$$

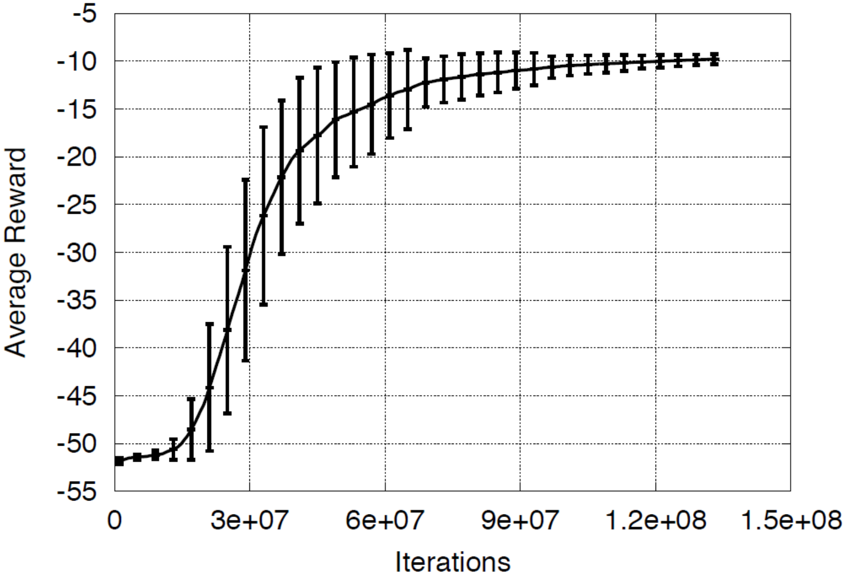
  end for

  end for

  return ϑ

Puck World Example





- Continuous actions exert small force on puck
- Puck is rewarded for getting close to target
- Target location is reset every 30 seconds
- Policy is trained using variant of MC policy gradient

Softmax Stochastic Policy

- Softmax policy is a very commonly used stochastic policy

$$\pi_\theta(a|s) = \frac{e^{f_\theta(s,a)}}{\sum_{a'} e^{f_\theta(s,a')}}$$

  - where $f_\vartheta(s,a)$ is the score function of a state-action pair parametrized by ϑ, which can be defined with domain knowledge

- The gradient of its log-likelihood

$$\begin{aligned}\frac{\partial \log \pi_\theta(a|s)}{\partial\theta} &= \frac{\partial f_\theta(s,a)}{\partial\theta} - \frac{1}{\sum_{a'} e^{f_\theta(s,a')}} \sum_{a''} e^{f_\theta(s,a'')} \frac{\partial f_\theta(s,a'')}{\partial\theta} \\ &= \frac{\partial f_\theta(s,a)}{\partial\theta} - \mathbb{E}_{a'\sim\pi_\theta(a'|s)}\left[\frac{\partial f_\theta(s,a')}{\partial\theta}\right]\end{aligned}$$

Softmax Stochastic Policy

- Softmax policy is a very commonly used stochastic policy

$$\pi_\theta(a|s) = \frac{e^{f_\theta(s,a)}}{\sum_{a'} e^{f_\theta(s,a')}}$$

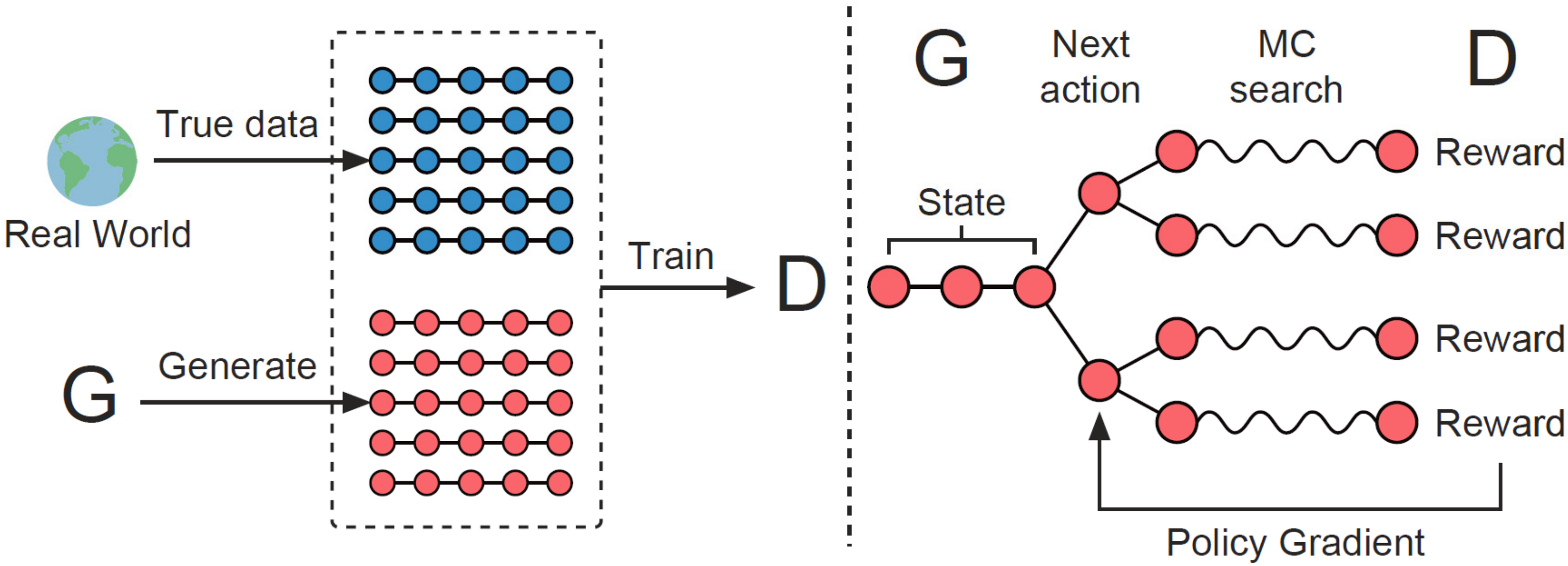
  - where $f_\vartheta(s,a)$ is the score function of a state-action pair parametrized by ϑ, which can be defined with domain knowledge

- For example, we define the linear score function

$$f_\theta(s,a) = \theta^\top x(s,a)$$

$$\begin{aligned}
\frac{\partial \log \pi_\theta(a|s)}{\partial \theta} &= \frac{\partial f_\theta(s,a)}{\partial \theta} - \mathbb{E}_{a' \sim \pi_\theta(a'|s)}\Big[\frac{\partial f_\theta(s,a')}{\partial \theta}\Big] \\
&= x(s,a) - \mathbb{E}_{a' \sim \pi_\theta(a'|s)}[x(s,a')]
\end{aligned}$$

Sequence Generation Example



- Generator is a reinforcement learning policy $G_\theta(y_t|Y_{1:t-1})$ of generating a sequence
  - Decide the next word (discrete action) to generate given the previous ones, implemented by softmax policy
  - Discriminator provides the reward (i.e. the probability of being true data) for the whole sequence
  - G is trained by MC policy gradient (REINFORCE)

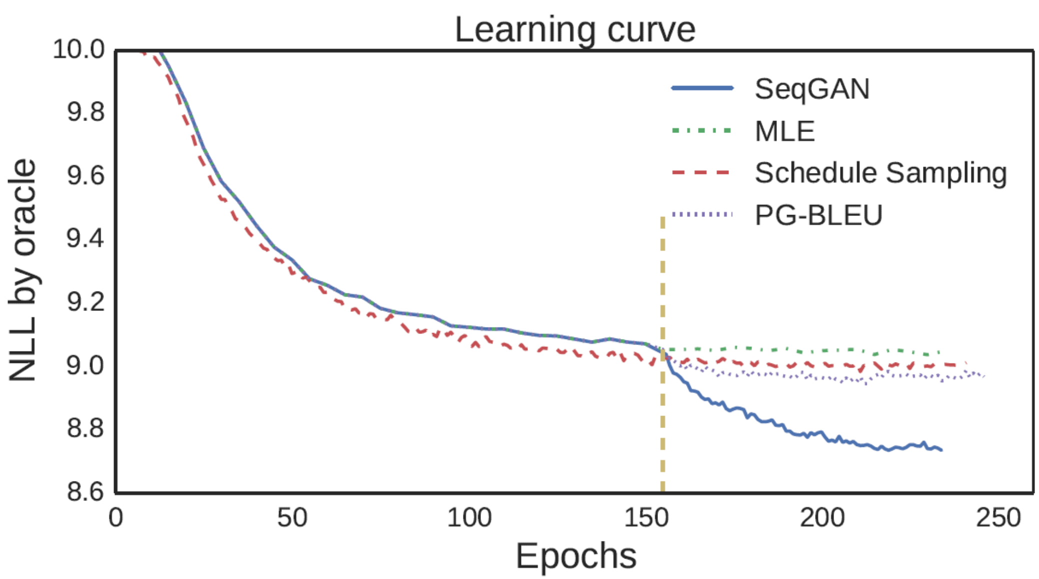
[Lantao Yu, Weinan Zhang, Jun Wang, Yong Yu. SeqGAN: Sequence Generative Adversarial Nets with Policy Gradient. AAAI 2017.]

Experiments on Synthetic Data

- Evaluation measure with Oracle

$$\mathrm{NLL}_{\mathrm{oracle}} = -\mathbb{E}_{Y_{1:T}\sim G_\theta}\left[\sum_{t=1}^{T}\log G_{\mathrm{oracle}}(y_t|Y_{1:t-1})\right]$$

Algorithm	Random	MLE	SS	PG-BLEU	SeqGAN
NLL	10.310	9.038	8.985	8.946	**8.736**
p-value	$< 10^{-6}$	$< 10^{-6}$	$< 10^{-6}$	$< 10^{-6}$	



[Lantao Yu, Weinan Zhang, Jun Wang, Yong Yu. SeqGAN: Sequence Generative Adversarial Nets with Policy Gradient. AAAI 2017.]

APPENDIX

Policy Gradient Theorem: Average Reward Setting

- Average reward objective

$$J(\pi) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\Big[r_1 + r_2 + \cdots + r_n | \pi\Big] = \sum_s d^\pi(s) \sum_a \pi(s,a) r(s,a)$$

$$Q^\pi(s,a) = \sum_{t=1}^{\infty} \mathbb{E}\Big[r_t - J(\pi) | s_0 = s, a_0 = a, \pi\Big]$$

$$\begin{aligned}
\frac{\partial V^\pi(s)}{\partial \theta} &\overset{\text{def}}{=} \frac{\partial}{\partial \theta} \sum_a \pi(s,a) Q^\pi(s,a), \quad \forall s \\
&= \sum_a \Big[\frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a) + \pi(s,a) \frac{\partial}{\partial \theta} Q^\pi(s,a)\Big] \\
&= \sum_a \Big[\frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a) + \pi(s,a) \frac{\partial}{\partial \theta}\Big(r(s,a) - J(\pi) + \sum_{s'} P^a_{ss'} V^\pi(s')\Big)\Big] \\
&= \sum_a \Big[\frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a) + \pi(s,a)\Big(- \frac{\partial J(\pi)}{\partial \theta} + \frac{\partial}{\partial \theta} \sum_{s'} P^a_{ss'} V^\pi(s')\Big)\Big] \\
\Rightarrow \frac{\partial J(\pi)}{\partial \theta} &= \sum_a \Big[\frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a) + \pi(s,a) \sum_{s'} P^a_{ss'} \frac{\partial V^\pi(s')}{\partial \theta}\Big] - \frac{\partial V^\pi(s)}{\partial \theta}
\end{aligned}$$

Please refer to Chapter 13 of Rich Sutton's Reinforcement Learning: An Introduction (2nd Edition)

APPENDIX

Policy Gradient Theorem: Average Reward Setting

- Average reward objective

$$\frac{\partial J(\pi)}{\partial\theta} = \sum_a \Big[\frac{\partial \pi(s,a)}{\partial\theta} Q^\pi(s,a) + \pi(s,a) \sum_{s'} P^a_{ss'} \frac{\partial V^\pi(s')}{\partial\theta}\Big] - \frac{\partial V^\pi(s)}{\partial\theta}$$

$$\sum_s d^\pi(s) \frac{\partial J(\pi)}{\partial\theta} = \sum_s d^\pi(s) \sum_a \frac{\partial \pi(s,a)}{\partial\theta} Q^\pi(s,a) + \sum_s d^\pi(s) \sum_a \pi(s,a) \sum_{s'} P^a_{ss'} \frac{\partial V^\pi(s')}{\partial\theta} - \sum_s d^\pi(s) \frac{\partial V^\pi(s)}{\partial\theta}$$

$$\sum_s d^\pi(s) \sum_a \pi(s,a) \sum_{s'} P^a_{ss'} \frac{\partial V^\pi(s')}{\partial\theta} = \sum_s \sum_a \sum_{s'} d^\pi(s) \pi(s,a) P^a_{ss'} \frac{\partial V^\pi(s')}{\partial\theta}$$

$$= \sum_s \sum_{s'} d^\pi(s) \Big(\sum_a \pi(s,a) P^a_{ss'} \Big) \frac{\partial V^\pi(s')}{\partial\theta} = \sum_s \sum_{s'} d^\pi(s) P_{ss'} \frac{\partial V^\pi(s')}{\partial\theta}$$

$$= \sum_{s'} \Big(\sum_s d^\pi(s) P_{ss'} \Big) \frac{\partial V^\pi(s')}{\partial\theta} = \sum_{s'} d^\pi(s') \frac{\partial V^\pi(s')}{\partial\theta}$$

$$\Rightarrow \sum_s d^\pi(s) \frac{\partial J(\pi)}{\partial\theta} = \sum_s d^\pi(s) \sum_a \frac{\partial \pi(s,a)}{\partial\theta} Q^\pi(s,a) + \sum_{s'} d^\pi(s') \frac{\partial V^\pi(s')}{\partial\theta} - \sum_s d^\pi(s) \frac{\partial V^\pi(s)}{\partial\theta}$$

$$\Rightarrow \frac{\partial J(\pi)}{\partial\theta} = \sum_s d^\pi(s) \sum_a \frac{\partial \pi(s,a)}{\partial\theta} Q^\pi(s,a)$$

Please refer to Chapter 13 of Rich Sutton's Reinforcement Learning: An Introduction (2^{nd} Edition)

APPENDIX

Policy gradient theorem: Start Value Setting

- Start state value objective

$$J(\pi) = \mathbb{E}\Big[\sum_{t=1}^{\infty}\gamma^{t-1}r_t\Big|s_0,\pi\Big]$$

$$Q^\pi(s,a) = \mathbb{E}\Big[\sum_{k=1}^{\infty}\gamma^{k-1}r_{t+k}\Big|s_t = s, a_t = a, \pi\Big]$$

$$\begin{aligned}
\frac{\partial V^\pi(s)}{\partial\theta} &\stackrel{\text{def}}{=} \frac{\partial}{\partial\theta}\sum_a \pi(s,a)Q^\pi(s,a), \quad \forall s \\
&= \sum_a \Big[\frac{\partial\pi(s,a)}{\partial\theta}Q^\pi(s,a) + \pi(s,a)\frac{\partial}{\partial\theta}Q^\pi(s,a)\Big] \\
&= \sum_a \Big[\frac{\partial\pi(s,a)}{\partial\theta}Q^\pi(s,a) + \pi(s,a)\frac{\partial}{\partial\theta}\Big(r(s,a) + \sum_{s'}\gamma P_{ss'}^a V^\pi(s')\Big)\Big] \\
&= \sum_a \frac{\partial\pi(s,a)}{\partial\theta}Q^\pi(s,a) + \sum_a \pi(s,a)\gamma\sum_{s'}P_{ss'}^a\frac{\partial V^\pi(s')}{\partial\theta}
\end{aligned}$$

Please refer to Chapter 13 of Rich Sutton's Reinforcement Learning: An Introduction (2nd Edition)

APPENDIX

Policy gradient theorem: Start Value Setting

- Start state value objective

$$\frac{\partial V^{\pi}(s)}{\partial \theta} = \sum_{a} \frac{\partial \pi(s,a)}{\partial \theta} Q^{\pi}(s,a) + \sum_{a} \pi(s,a) \gamma \sum_{s_1} P^{a}_{ss_1} \frac{\partial V^{\pi}(s_1)}{\partial \theta}$$

$$\sum_{a} \frac{\partial \pi(s,a)}{\partial \theta} Q^{\pi}(s,a) = \gamma^0 Pr(s \to s, 0, \pi) \sum_{a} \frac{\partial \pi(s,a)}{\partial \theta} Q^{\pi}(s,a)$$

$$\begin{aligned}
\sum_{a} \pi(s,a) \gamma \sum_{s_1} P^{a}_{ss_1} \frac{\partial V^{\pi}(s_1)}{\partial \theta} &= \sum_{s_1} \sum_{a} \pi(s,a) \gamma P^{a}_{ss_1} \frac{\partial V^{\pi}(s_1)}{\partial \theta} \\
&= \sum_{s_1} \gamma P_{ss_1} \frac{\partial V^{\pi}(s_1)}{\partial \theta} = \gamma^1 \sum_{s_1} Pr(s \to s_1, 1, \pi) \frac{\partial V^{\pi}(s_1)}{\partial \theta}
\end{aligned}$$

$$\frac{\partial V^{\pi}(s_1)}{\partial \theta} = \sum_{a} \frac{\partial \pi(s,a)}{\partial \theta} Q^{\pi}(s,a) + \gamma^1 \sum_{s_2} Pr(s_1 \to s_2, 1, \pi) \frac{\partial V^{\pi}(s_2)}{\partial \theta}$$

Please refer to Chapter 13 of Rich Sutton's Reinforcement Learning: An Introduction (2nd Edition)

APPENDIX

Policy gradient theorem: Start Value Setting

- Start state value objective

$$
\begin{aligned}
\frac{\partial V^\pi(s)}{\partial \theta} &= \gamma^0 Pr(s \to s, 0, \pi) \sum_a \frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a) + \gamma^1 \sum_{s_1} Pr(s \to s_1, 1, \pi) \sum_a \frac{\partial \pi(s_1,a)}{\partial \theta} Q^\pi(s_1,a) \\
&\quad + \gamma^2 \sum_{s_1} Pr(s \to s_1, 1, \pi) \sum_{s_2} Pr(s_1 \to s_2, 1, \pi) \frac{\partial V^\pi(s_2)}{\partial \theta} \\
&= \gamma^0 Pr(s \to s, 0, \pi) \sum_a \frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a) + \gamma^1 \sum_{s_1} Pr(s \to s_1, 1, \pi) \sum_a \frac{\partial \pi(s_1,a)}{\partial \theta} Q^\pi(s_1,a) \\
&\quad + \gamma^2 \sum_{s_2} Pr(s \to s_2, 2, \pi) \frac{\partial V^\pi(s_2)}{\partial \theta} \\
&= \sum_{k=0}^{\infty} \sum_x \gamma^k Pr(s \to x, k, \pi) \sum_a \frac{\partial \pi(x,a)}{\partial \theta} Q^\pi(x,a) = \sum_x \sum_{k=0}^{\infty} \gamma^k Pr(s \to x, k, \pi) \sum_a \frac{\partial \pi(x,a)}{\partial \theta} Q^\pi(x,a)
\end{aligned}
$$

$$
\Rightarrow \frac{\partial J(\pi)}{\partial \theta} = \frac{\partial V^\pi(s_0)}{\partial \theta} = \sum_s \sum_{k=0}^{\infty} \gamma^k Pr(s_0 \to s, k, \pi) \sum_a \frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a) = \sum_s d^\pi(s) \sum_a \frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a)
$$

Please refer to Chapter 13 of Rich Sutton’s Reinforcement Learning: An Introduction (2nd Edition)

Problems of REINFORCE

- Episodic data tasks
  - Normally the task should have terminal state so the cumulated discounted rewards can be directly calculated for REINFORCE
- Low data efficiency
  - In practice, REINFORCE needs a large amount of data for training
- High training variance (most important shortage)
  - Sampling the value from one or several episode includes very high variance

Actor-Critic

- Idea of actor-critic
  - REINFORCE policy gradient uses Monte Carlo sampling to directly estimate the value of (s_t, a_t) as G_t
  - Why not to build a trainable value function Q_ϕ to perform such an estimation?
- Actor and Critic

Actor $\pi_\theta(a|s)$

The policy of taking actions that learns to make critic satisfied

Critic $Q_\phi(s, a)$

The value function that learns to accurately estimate the action value of actor policy

Actor-Critic Training

- Critic $Q_\phi(s,a)$
  - learns to accurately estimate the action value of the current actor policy

$$Q_\phi(s,a) \simeq r(s,a) + \gamma \mathbb{E}_{s' \sim p(s'|s,a), a' \sim \pi_\theta(a'|s')}[Q_\phi(s',a')]$$

- Actor $\pi_\theta(a|s)$
  - learns to take actions that make critic satisfied

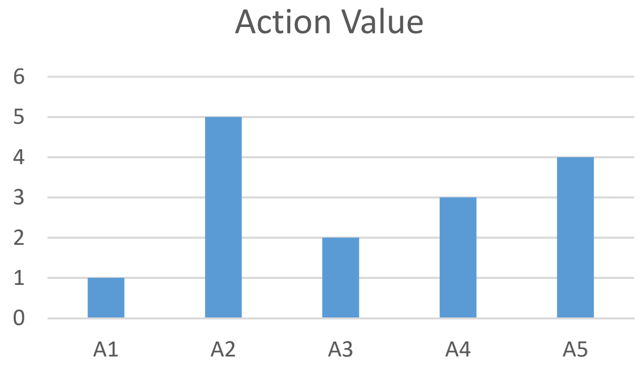
$$J(\theta) = \mathbb{E}_{s \sim p, \pi_\theta}[\pi_\theta(a|s) Q_\phi(s,a)]$$

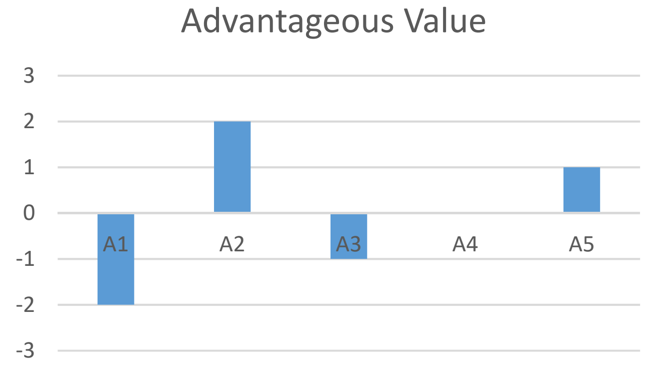
$$\frac{\partial J(\theta)}{\partial \theta} = \mathbb{E}_{\pi_\theta}\Big[\frac{\partial \log \pi_\theta(a|s)}{\partial \theta} Q_\phi(s,a)\Big]$$

A2C: Advantageous Actor-Critic

- Idea: to normalize the critic score by subtracting a baseline function
  - More informative guidance: lower the probability of bad actions and higher the one of good actions
  - Further lower variance
- Advantageous function

$$A^{\pi}(s, a) = Q^{\pi}(s, a) - V^{\pi}(s)$$





A2C: Advantageous Actor-Critic

- Action value and state value functions

$$\begin{aligned} Q^{\pi}(s,a) &= r(s,a) + \gamma \mathbb{E}_{s' \sim p(s'|s,a), a' \sim \pi_\theta(a'|s')}[Q_\phi(s',a')] \\ &= r(s,a) + \gamma \mathbb{E}_{s' \sim p(s'|s,a)}[V^{\pi}(s')] \end{aligned}$$

- Thus we only need to fit state value function to fit the advantageous function

$$\begin{aligned} A^{\pi}(s,a) &= Q^{\pi}(s,a) - V^{\pi}(s) \\ &= r(s,a) + \gamma \mathbb{E}_{s' \sim p(s'|s,a)}[V^{\pi}(s') - V^{\pi}(s)] \\ &\simeq r(s,a) + \gamma (V^{\pi}(s') - V^{\pi}(s)) \end{aligned}$$

Sample the next state s'

More Explanations and Experiments of A2C

- https://medium.freecodecamp.org/an-intro-to-advantage-actor-critic-methods-lets-play-sonic-the-hedgehog-86d6240171d

- https://medium.com/m/global-identity?redirectUrl=https%3A%2F%2Fhackernoon.com%2Fintuitive-rl-intro-to-advantage-actor-critic-a2c-4ff545978752