Approximation Methods in Reinforcement Learning

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http://wnzhang.net/teaching/cs420/index.html
Our course on RL is mainly based on the materials from these masters.

Prof. Richard Sutton
- University of Alberta, Canada
- http://incompleteideas.net/sutton/index.html
- Reinforcement Learning: An Introduction (2nd edition)

Dr. David Silver
- Google DeepMind and UCL, UK
- http://www0.cs.ucl.ac.uk/staff/d.silver/web/Home.html
- UCL Reinforcement Learning Course
- http://www0.cs.ucl.ac.uk/staff/d.silver/web/Teaching.html

Prof. Andrew Ng
- Stanford University, US
- http://www.andrewng.org/
- Machine Learning (CS229) Lecture Notes 12: RL
Last Lecture

• Model-based dynamic programming
  • Value iteration \( V(s) = R(s) + \max_{a \in A} \gamma \sum_{s' \in S} P_{sa}(s') V(s') \)
  • Policy iteration \( \pi(s) = \arg \max_{a \in A} \sum_{s' \in S} P_{sa}(s') V(s') \)

• Model-free reinforcement learning
  • On-policy MC \( V(s_t) \leftarrow V(s_t) + \alpha(G_t - V(s_t)) \)
  • On-policy TD \( V(s_t) \leftarrow V(s_t) + \alpha(r_{t+1} + \gamma V(s_{t+1}) - V(s_t)) \)
  • On-policy TD SARSA
    \[ Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_{t+1} + \gamma Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t)) \]
  • Off-policy TD Q-learning
    \[ Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_{t+1} + \gamma \max_{a'} Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t)) \]
Key Problem to Solve in This Lecture

• In all previous models, we have created a lookup table to maintain a variable $V(s)$ for each state or $Q(s,a)$ for each state-action

• What if we have a large MDP, i.e.
  • the state or state-action space is too large
  • or the state or action space is continuous
to maintain $V(s)$ for each state or $Q(s,a)$ for each state-action?
• For example
  • Game of Go ($10^{170}$ states)
  • Helicopter, autonomous car (continuous state space)
Content

• Solutions for large MDPs
  • Discretize or bucketize states/actions
  • Build parametric value function approximation

• Policy gradient

• Deep reinforcement learning and multi-agent RL
Content

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Discretization Continuous MDP

• For a continuous-state MDP, we can discretize the state space
  • For example, if we have 2D states \((s_1, s_2)\), we can use a grid to discretize the state space
  • The discrete state \(\bar{s}\)
  • The discretized MDP:
    \[
    (\bar{S}, A, \{P_{\bar{s}a}\}, \gamma, R)
    \]
  • Then solve this MDP with any previous solutions
Bucketize Large Discrete MDP

• For a large discrete-state MDP, we can bucketize the states to downsample the states
  • To use domain knowledge to merge similar discrete states
    • For example, clustering using state features extracted from domain knowledge
Discretization/Bucketization

• Pros
  • Straightforward and off-the-shelf
  • Efficient
  • Can work well for many problems

• Cons
  • A fairly naïve representation for $V$
  • Assumes a constant value over each discretized cell
  • Curse of dimensionality

\[
S = \mathbb{R}^n \Rightarrow \tilde{S} = \{1, \ldots, k\}^n
\]
Parametric Value Function Approximation

• Create parametric (thus learnable) functions to approximate the value function

\[ V_\theta(s) \simeq V^\pi(s) \]
\[ Q_\theta(s, a) \simeq Q^\pi(s, a) \]

• \( \theta \) is the parameters of the approximation function, which can be updated by reinforcement learning

• Generalize from seen states to unseen states
Main Types of Value Function Approx.

$$V_\theta(s)$$  

$$Q_\theta(s, a)$$

Many function approximations
- (Generalized) linear model
- Neural network
- Decision tree
- Nearest neighbor
- Fourier / wavelet bases

Differentiable functions
- (Generalized) linear model
- Neural network

We assume the model is suitable to be trained for non-stationary, non-iid data
Value Function Approx. by SGD

• Goal: find parameter vector \( \theta \) minimizing mean-squared error between approximate value function \( V_\theta(s) \) and true value \( V^\pi(s) \)

\[
J(\theta) = \mathbb{E}_\pi \left[ \frac{1}{2} (V^\pi(s) - V_\theta(s))^2 \right]
\]

• Gradient to minimize the error

\[
- \frac{\partial J(\theta)}{\partial \theta} = \mathbb{E}_\pi \left[ (V^\pi(s) - V_\theta(s)) \frac{\partial V_\theta(s)}{\partial \theta} \right]
\]

• Stochastic gradient descent on one sample

\[
\theta \leftarrow \theta - \alpha \frac{\partial J(\theta)}{\partial \theta}
\]

\[
= \theta + \alpha (V^\pi(s) - V_\theta(s)) \frac{\partial V_\theta(s)}{\partial \theta}
\]
Featurize the State

• Represent state by a feature vector
  
  \[ x(s) = \begin{bmatrix} 
  x_1(s) \\ 
  \vdots \\ 
  x_k(s) 
  \end{bmatrix} \]

• For example of a helicopter
  
  • 3D location
  • 3D speed (differentiation of location)
  • 3D acceleration (differentiation of speed)
Linear Value Function Approximation

• Represent value function by a linear combination of features

\[ V_{\theta}(s) = \theta^\top x(s) \]

• Objective function is quadratic in parameters \( \theta \)

\[ J(\theta) = \mathbb{E}_\pi \left[ \frac{1}{2} (V^\pi(s) - \theta^\top x(s))^2 \right] \]

• Thus stochastic gradient descent converges on global optimum

\[ \theta \leftarrow \theta - \alpha \frac{\partial J(\theta)}{\partial \theta} \]

\[ = \theta + \alpha (V^\pi(s) - V_{\theta}(s)) x(s) \]
Monte-Carlo with Value Function Approx.

\[ \theta \leftarrow \theta + \alpha(V^\pi(s) - V_\theta(s))x(s) \]

• Now we specify the target value function \( V^\pi(s) \)
• We can apply supervised learning to “training data”

\[ \langle s_1, G_1 \rangle, \langle s_2, G_2 \rangle, \ldots, \langle s_T, G_T \rangle \]

• For each data instance \(<s_t, G_t>\)

\[ \theta \leftarrow \theta + \alpha(G_t - V_\theta(s))x(s_t) \]

• MC evaluation at least converges to a local optimum
  • In linear case it converges to a global optimum
TD Learning with Value Function Approx.

\[ \theta \leftarrow \theta + \alpha(V^\pi(s) - V_\theta(s))x(s) \]

- TD target \( r_{t+1} + \gamma V_\theta(s_{t+1}) \) is a biased sample of true target value \( V^\pi(s_t) \)
- Supervised learning from “training data”
  \[ \langle s_1, r_2 + \gamma V_\theta(s_2) \rangle, \langle s_2, r_3 + \gamma V_\theta(s_3) \rangle, \ldots, \langle s_T, r_T \rangle \]
- For each data instance \( \langle s_t, r_{t+1} + \gamma V_\theta(s_{t+1}) \rangle \)
  \[ \theta \leftarrow \theta + \alpha(r_{t+1} + \gamma V_\theta(s_{t+1}) - V_\theta(s))x(s_t) \]
- Linear TD converges (close) to global optimum
Action-Value Function Approximation

• Approximate the action-value function

\[ Q_\theta(s, a) \simeq Q^\pi(s, a) \]

• Minimize mean squared error

\[ J(\theta) = \mathbb{E}_\pi \left[ \frac{1}{2} (Q^\pi(s, a) - Q_\theta(s, a))^2 \right] \]

• Stochastic gradient descent on one sample

\[
\theta \leftarrow \theta - \alpha \frac{\partial J(\theta)}{\partial \theta} \\
= \theta + \alpha(Q^\pi(s, a) - Q_\theta(s, a)) \frac{\partial Q_\theta(s, a)}{\partial \theta}
\]
Linear Action-Value Function Approx.

• Represent state-action pair by a feature vector

\[ x(s, a) = \begin{bmatrix} x_1(s, a) \\ \vdots \\ x_k(s, a) \end{bmatrix} \]

• Parametric Q function, e.g., the linear case

\[ Q_\theta(s, a) = \theta^\top x(s, a) \]

• Stochastic gradient descent update

\[
\theta \leftarrow \theta - \alpha \frac{\partial J(\theta)}{\partial \theta} \\
= \theta + \alpha (Q^\pi(s, a) - \theta^\top x(s, a)) x(s, a)
\]
TD Learning with Value Function Approx.

\[ \theta \leftarrow \theta + \alpha (Q^\pi(s, a) - Q_\theta(s, a)) \frac{\partial Q_\theta(s, a)}{\partial \theta} \]

- For MC, the target is the return \( G_t \)
  \[ \theta \leftarrow \theta + \alpha (G_t - Q_\theta(s, a)) \frac{\partial Q_\theta(s, a)}{\partial \theta} \]

- For TD, the target is \( r_{t+1} + \gamma Q_\theta(s_{t+1}, a_{t+1}) \)
  \[ \theta \leftarrow \theta + \alpha (r_{t+1} + \gamma Q_\theta(s_{t+1}, a_{t+1}) - Q_\theta(s, a)) \frac{\partial Q_\theta(s, a)}{\partial \theta} \]
Control with Value Function Approx.

- Policy evaluation: approximately policy evaluation \( Q_\theta \simeq Q^\pi \)
- Policy improvement: \( \varepsilon \)-greedy policy improvement
NOTE of TD Update

• For TD(0), the TD target is

  • State value

    \[
    \theta \leftarrow \theta + \alpha (V^\pi(s_t) - V_\theta(s_t)) \frac{\partial V_\theta(s_t)}{\partial \theta}
    \]

    \[
    = \theta + \alpha (r_{t+1} + \gamma V_\theta(s_{t+1}) - V_\theta(s)) \frac{\partial V_\theta(s_t)}{\partial \theta}
    \]

  • Action value

    \[
    \theta \leftarrow \theta + \alpha (Q^\pi(s, a) - Q_\theta(s, a)) \frac{\partial Q_\theta(s, a)}{\partial \theta}
    \]

    \[
    = \theta + \alpha (r_{t+1} + \gamma Q_\theta(s_{t+1}, a_{t+1}) - Q_\theta(s, a)) \frac{\partial Q_\theta(s, a)}{\partial \theta}
    \]

• Although \( \theta \) is in the TD target, we don’t calculate gradient from the target. Think about why.
Case Study: Mountain Car

The gravity is stronger than the car’s engine

Cost-to-go function
Case Study: Mountain Car

Mountain Car
Steps per episode
log scale
averaged over 100 runs

Mountain Car
Steps per episode
averaged over first 50 episodes and 100 runs
Deep Q-Network (DQN)

Deep Q-Network (DQN)

• Implement Q function with deep neural network

Deep Q-Network (DQN)

• The loss function of Q-learning update at iteration $i$

$$L_i(\theta_i) = \mathbb{E}_{(s,a,r,s') \sim U(D)} \left[ (r + \gamma \max_{a'} Q(s', a'; \theta^-_i) - Q(s, a; \theta_i))^2 \right]$$

  - $\theta_i$ are the network parameters to be updated at iteration $i$
    - Updated with standard back-propagation algorithms
  - $\theta^-_i$ are the target network parameters
    - Only updated with $\theta_i$ for every $C$ steps
  - $(s,a,r,s') \sim U(D)$: the samples are uniformly drawn from the experience pool $D$
    - Thus to avoid the overfitting to the recent experiences

Content

- Solutions for large MDPs
  - Discretize or bucketize states/actions
  - Build parametric value function approximation

- Policy gradient

- Deep reinforcement learning and multi-agent RL
Parametric Policy

• We can parametrize the policy

\[ \pi_\theta(a|s) \]

which could be deterministic

\[ a = \pi_\theta(s) \]

or stochastic

\[ \pi_\theta(a|s) = P(a|s; \theta) \]

• \( \theta \) is the parameters of the policy
• Generalize from seen states to unseen states
• We focus on model-free reinforcement learning
Policy-based RL

• Advantages
  • Better convergence properties
  • Effective in high-dimensional or continuous action spaces
    • No.1 reason: for value function, you have to take a max operation
  • Can learn stochastic polices

• Disadvantages
  • Typically converge to a local rather than global optimum
  • Evaluating a policy is typically inefficient and of high variance
Policy Gradient

• For stochastic policy \( \pi_\theta(a|s) = P(a|s; \theta) \)

• Intuition
  • lower the probability of the action that leads to low value/reward
  • higher the probability of the action that leads to high value/reward

• A 5-action example

1. Initialize \( \theta \)

Action Probability

2. Take action A2
   Observe positive reward

3. Update \( \theta \) by policy gradient

Action Probability

4. Take action A3
   Observe negative reward

5. Update \( \theta \) by policy gradient
Policy Gradient in One-Step MDPs

• Consider a simple class of one-step MDPs
  • Starting in state \( s \sim d(s) \)
  • Terminating after one time-step with reward \( r_{s a} \)

• Policy expected value

\[
J(\theta) = \mathbb{E}_{\pi_{\theta}}[r] = \sum_{s \in S} d(s) \sum_{a \in A} \pi_{\theta}(a|s)r_{sa}
\]

\[
\frac{\partial J(\theta)}{\partial \theta} = \sum_{s \in S} d(s) \sum_{a \in A} \frac{\partial \pi_{\theta}(a|s)}{\partial \theta} r_{sa}
\]
Likelihood Ratio

- Likelihood ratios exploit the following identity
  \[
  \frac{\partial \pi_\theta(a|s)}{\partial \theta} = \pi_\theta(a|s) \frac{1}{\pi_\theta(a|s)} \frac{\partial \pi_\theta(a|s)}{\partial \theta}
  = \pi_\theta(a|s) \frac{\partial \log \pi_\theta(a|s)}{\partial \theta}
  \]

- Thus the policy’s expected value
  \[
  J(\theta) = \mathbb{E}_{\pi_\theta}[r] = \sum_{s \in S} d(s) \sum_{a \in A} \pi_\theta(a|s) r_{sa},
  \]
  \[
  \frac{\partial J(\theta)}{\partial \theta} = \sum_{s \in S} d(s) \sum_{a \in A} \frac{\partial \pi_\theta(a|s)}{\partial \theta} r_{sa}
  = \sum_{s \in S} d(s) \sum_{a \in A} \pi_\theta(a|s) \frac{\partial \log \pi_\theta(a|s)}{\partial \theta} r_{sa}
  = \mathbb{E}_{\pi_\theta} \left[ \frac{\partial \log \pi_\theta(a|s)}{\partial \theta} r_{sa} \right]
  \]
  This can be approximated by sampling state s from d(s) and action a from \(\pi_\theta\).
Policy Gradient Theorem

- The policy gradient theorem generalizes the likelihood ratio approach to multi-step MDPs
  - Replaces instantaneous reward $r_{sa}$ with long-term value $Q^{\pi_\theta}(s, a)$
- Policy gradient theorem applies to
  - start state objective $J_1$, average reward objective $J_{avR}$, and average value objective $J_{avV}$
- Theorem
  - For any differentiable policy $\pi_\theta(a|s)$, for any of policy objective function $J = J_1, J_{avR}, J_{avV}$, the policy gradient is
    \[
    \frac{\partial J(\theta)}{\partial \theta} = \mathbb{E}_{\pi_\theta} \left[ \frac{\partial \log \pi_\theta(a|s)}{\partial \theta} Q^{\pi_\theta}(s, a) \right]
    \]

Please refer to appendix of the slides for detailed proofs
Monte-Carlo Policy Gradient (REINFORCE)

- Update parameters by stochastic gradient ascent
- Using policy gradient theorem
- Using return $G_t$ as an unbiased sample of $Q^{\pi_\theta}(s, a)$

$$\Delta \theta_t = \alpha \frac{\partial \log \pi_\theta(a_t|s_t)}{\partial \theta} G_t$$

- REINFORCE Algorithm
  
  Initialize $\theta$ arbitrarily
  
  for each episode $\{s_1, a_1, r_2, \ldots, s_{T-1}, a_{T-1}, r_T\} \sim \pi_\theta$ do
    
    for $t=1$ to $T-1$ do
      
      $\theta \leftarrow \theta + \alpha \frac{\partial}{\partial \theta} \log \pi_\theta(a_t|s_t) G_t$
    
    end for
  
  end for

  return $\theta$
Puck World Example

- Continuous actions exert small force on puck
- Puck is rewarded for getting close to target
- Target location is reset every 30 seconds
- Policy is trained using variant of MC policy gradient
Softmax Stochastic Policy

- Softmax policy is a very commonly used stochastic policy

\[ \pi_\theta(a|s) = \frac{e^{f_\theta(s,a)}}{\sum_{a'} e^{f_\theta(s,a')}} \]

- where \( f_\vartheta(s,a) \) is the score function of a state-action pair parametrized by \( \vartheta \), which can be defined with domain knowledge

- The gradient of its log-likelihood

\[
\frac{\partial \log \pi_\theta(a|s)}{\partial \theta} = \frac{\partial f_\theta(s,a)}{\partial \theta} - \frac{1}{\sum_{a'} e^{f_\theta(s,a')}} \sum_{a''} e^{f_\theta(s,a'')} \frac{\partial f_\theta(s,a'')}{\partial \theta} \\
= \frac{\partial f_\theta(s,a)}{\partial \theta} - \mathbb{E}_{a' \sim \pi_\theta(a'|s)} \left[ \frac{\partial f_\theta(s,a')}{\partial \theta} \right]
\]
Softmax Stochastic Policy

- Softmax policy is a very commonly used stochastic policy

\[ \pi_\theta(a|s) = \frac{e^{f_\theta(s,a)}}{\sum_{a'} e^{f_\theta(s,a')}} \]

- where \( f_\theta(s,a) \) is the score function of a state-action pair parametrized by \( \theta \), which can be defined with domain knowledge

- For example, we define the linear score function

\[ f_\theta(s, a) = \theta^\top x(s, a) \]

\[
\frac{\partial \log \pi_\theta(a|s)}{\partial \theta} = \frac{\partial f_\theta(s, a)}{\partial \theta} - \mathbb{E}_{a' \sim \pi_\theta(a'|s)} \left[ \frac{\partial f_\theta(s, a')}{\partial \theta} \right] \\
= x(s, a) - \mathbb{E}_{a' \sim \pi_\theta(a'|s)}[x(s, a')]\]
Sequence Generation Example

- Generator is a reinforcement learning policy $G_{\theta}(y_t|Y_{1:t-1})$ of generating a sequence
  - Decide the next word (discrete action) to generate given the previous ones, implemented by softmax policy
  - Discriminator provides the reward (i.e. the probability of being true data) for the whole sequence
  - G is trained by MC policy gradient (REINFORCE)

Experiments on Synthetic Data

• Evaluation measure with Oracle

\[
\text{NLL}_\text{oracle} = -\mathbb{E}_{Y_1:T \sim G_\theta} \left[ \sum_{t=1}^{T} \log G_\text{oracle}(y_t | Y_{1:t-1}) \right]
\]

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<tr>
<th>Algorithm</th>
<th>Random</th>
<th>MLE</th>
<th>SS</th>
<th>PG-BLEU</th>
<th>SeqGAN</th>
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<tbody>
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<td>NLL</td>
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<td>9.038</td>
<td>8.985</td>
<td>8.946</td>
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Content

• Solutions for large MDPs
  • Discretize or bucketize states/actions
  • Build parametric value function approximation

• Policy gradient

• Deep reinforcement learning and multi-agent RL
  • By our invited speakers Yuhuai Wu, Shixiang Gu and Ying Wen
APPENDIX
Policy Gradient Theorem: Average Reward Setting

- Average reward objective

\[
J(\pi) = \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left[ r_1 + r_2 + \cdots + r_n \mid \pi \right] = \sum_s d^\pi(s) \sum_a \pi(s,a) r(s,a)
\]

\[
Q^\pi(s,a) = \sum_{t=1}^{\infty} \mathbb{E} \left[ r_t - J(\pi) \mid s_0 = s, a_0 = a, \pi \right]
\]

\[
\frac{\partial V^\pi(s)}{\partial \theta} \overset{\text{def}}{=} \frac{\partial}{\partial \theta} \sum_a \pi(s,a) Q^\pi(s,a), \quad \forall s
\]

\[
= \sum_a \left[ \frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a) + \pi(s,a) \frac{\partial}{\partial \theta} Q^\pi(s,a) \right]
\]

\[
= \sum_a \left[ \frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a) + \pi(s,a) \frac{\partial}{\partial \theta} \left( r(s,a) - J(\pi) + \sum_{s'} P_{ss'}^a V^\pi(s') \right) \right]
\]

\[
= \sum_a \left[ \frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a) + \pi(s,a) \left( - \frac{\partial J(\pi)}{\partial \theta} + \frac{\partial}{\partial \theta} \sum_{s'} P_{ss'}^a V^\pi(s') \right) \right]
\]

\[
\Rightarrow \frac{\partial J(\pi)}{\partial \theta} = \sum_a \left[ \frac{\partial \pi(s,a)}{\partial \theta} Q^\pi(s,a) + \pi(s,a) \sum_{s'} P_{ss'}^a \frac{\partial V^\pi(s')}{\partial \theta} \right] - \frac{\partial V^\pi(s)}{\partial \theta}
\]

Policy Gradient Theorem: Average Reward Setting

- Average reward objective

\[
\frac{\partial J(\pi)}{\partial \theta} = \sum_a \left[ \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a) + \pi(s, a) \sum_{s'} P^a_{ss'} \frac{\partial V^\pi(s')}{\partial \theta} \right] - \frac{\partial V^\pi(s)}{\partial \theta}
\]

\[
\sum_s d^\pi(s) \frac{\partial J(\pi)}{\partial \theta} = \sum_s d^\pi(s) \sum_a \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a) + \sum_s d^\pi(s) \sum_a \pi(s, a) \sum_{s'} P^a_{ss'} \frac{\partial V^\pi(s')}{\partial \theta} - \sum_s d^\pi(s) \frac{\partial V^\pi(s)}{\partial \theta}
\]

\[
\sum_s d^\pi(s) \sum_a \pi(s, a) \sum_{s'} P^a_{ss'} \frac{\partial V^\pi(s')}{\partial \theta} = \sum_s \sum_s \sum_{s'} d^\pi(s) \pi(s, a) P^a_{ss'} \frac{\partial V^\pi(s')}{\partial \theta}
\]

\[
= \sum_s \sum_{s'} \left( \sum_s d^\pi(s) P_{ss'} \right) \frac{\partial V^\pi(s')}{\partial \theta} = \sum_{s'} \sum_s d^\pi(s) P_{ss'} \frac{\partial V^\pi(s')}{\partial \theta}
\]

\[
\Rightarrow \sum_s d^\pi(s) \frac{\partial J(\pi)}{\partial \theta} = \sum_s d^\pi(s) \sum_a \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a) + \sum_{s'} d^\pi(s') \frac{\partial V^\pi(s')}{\partial \theta} - \sum_s d^\pi(s) \frac{\partial V^\pi(s)}{\partial \theta}
\]

\[
\Rightarrow \frac{\partial J(\pi)}{\partial \theta} = \sum_s d^\pi(s) \sum_a \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a)
\]
APPENDIX

Policy gradient theorem: Start Value Setting

• Start state value objective

\[ J(\pi) = \mathbb{E} \left[ \sum_{t=1}^{\infty} \gamma^{t-1} r_t \mid s_0, \pi \right] \]

\[ Q^{\pi}(s, a) = \mathbb{E} \left[ \sum_{k=1}^{\infty} \gamma^{k-1} r_{t+k} \mid s_t = s, a_t = a, \pi \right] \]

\[ \frac{\partial V^{\pi}(s)}{\partial \theta} \overset{\text{def}}{=} \frac{\partial}{\partial \theta} \sum_a \pi(s, a) Q^{\pi}(s, a), \quad \forall s \]

\[ = \sum_a \left[ \frac{\partial \pi(s, a)}{\partial \theta} Q^{\pi}(s, a) + \pi(s, a) \frac{\partial}{\partial \theta} Q^{\pi}(s, a) \right] \]

\[ = \sum_a \left[ \frac{\partial \pi(s, a)}{\partial \theta} Q^{\pi}(s, a) + \pi(s, a) \frac{\partial}{\partial \theta} \left( r(s, a) + \sum_{s'} \gamma P_{ss'}^{a} V^{\pi}(s') \right) \right] \]

\[ = \sum_a \frac{\partial \pi(s, a)}{\partial \theta} Q^{\pi}(s, a) + \sum_a \pi(s, a) \gamma \sum_{s'} P_{ss'}^{a} \frac{\partial V^{\pi}(s')}{\partial \theta} \]

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Policy gradient theorem: Start Value Setting

• Start state value objective

\[
\frac{\partial V^\pi(s)}{\partial \theta} = \sum_a \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a) + \sum_a \pi(s, a) \gamma \sum_{s_1} P^a_{ss_1} \frac{\partial V^\pi(s_1)}{\partial \theta}
\]

\[
\sum_a \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a) = \gamma^0 Pr(s \rightarrow s, 0, \pi) \sum_a \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a)
\]

\[
\sum_a \pi(s, a) \gamma \sum_{s_1} P^a_{ss_1} \frac{\partial V^\pi(s_1)}{\partial \theta} = \sum_{s_1} \sum_a \pi(s, a) \gamma P^a_{ss_1} \frac{\partial V^\pi(s_1)}{\partial \theta}
\]

\[
= \sum_{s_1} \gamma P^a_{ss_1} \frac{\partial V^\pi(s_1)}{\partial \theta} = \gamma^1 \sum_{s_1} Pr(s \rightarrow s_1, 1, \pi) \frac{\partial V^\pi(s_1)}{\partial \theta}
\]

\[
\frac{\partial V^\pi(s_1)}{\partial \theta} = \sum_a \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a) + \gamma^1 \sum_{s_2} Pr(s_1 \rightarrow s_2, 1, \pi) \frac{\partial V^\pi(s_2)}{\partial \theta}
\]

APPENDIX

Policy gradient theorem: Start Value Setting

• Start state value objective

\[
\frac{\partial V^\pi(s)}{\partial \theta} = \gamma^0 Pr(s \rightarrow s, 0, \pi) \sum_a \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a) + \gamma^1 \sum_{s_1} Pr(s \rightarrow s_1, 1, \pi) \sum_a \frac{\partial \pi(s_1, a)}{\partial \theta} Q^\pi(s_1, a) \\
+ \gamma^2 \sum_{s_1} Pr(s \rightarrow s_1, 1, \pi) \sum_{s_2} Pr(s_1 \rightarrow s_2, 1, \pi) \frac{\partial V^\pi(s_2)}{\partial \theta} \\
= \gamma^0 Pr(s \rightarrow s, 0, \pi) \sum_a \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a) + \gamma^1 \sum_{s_1} Pr(s \rightarrow s_1, 1, \pi) \sum_a \frac{\partial \pi(s_1, a)}{\partial \theta} Q^\pi(s_1, a) \\
+ \gamma^2 \sum_{s_2} Pr(s \rightarrow s_2, 2, \pi) \frac{\partial V^\pi(s_2)}{\partial \theta} \\
= \sum_{k=0}^{\infty} \sum_x \gamma^k Pr(s \rightarrow x, k, \pi) \sum_a \frac{\partial \pi(x, a)}{\partial \theta} Q^\pi(x, a) = \sum_x \sum_{k=0}^{\infty} \gamma^k Pr(s \rightarrow x, k, \pi) \sum_a \frac{\partial \pi(x, a)}{\partial \theta} Q^\pi(x, a) \\
\Rightarrow \frac{\partial J(\pi)}{\partial \theta} = \frac{\partial V^\pi(s_0)}{\partial \theta} = \sum_{s} \sum_{k=0}^{\infty} \gamma^k Pr(s_0 \rightarrow s, k, \pi) \sum_a \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a) = \sum_s d^\pi(s) \sum_a \frac{\partial \pi(s, a)}{\partial \theta} Q^\pi(s, a)
\]